New Considerations about the Switch Criterion for Defined Contribution Pension Scheme

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Abstract

In this work we have done so, more comprehensive analysis of the model proposed by the Authors Arts and Vigna "A switch criterion for defined contribution pension schemes" for the purpose of considering whether the most realistic market conditions in which to operate, the model itself may allow to obtain better performance. We kept the basic structure of jointly examining the accumulation phase and the distribution phase by inserting the correlation between financial assets. We have also identified a new method for calculating the reserves than the original model and finally we have extended the analysis of the distribution phase including the assumption of income drawdown option with bequest payable to the member's relative. From the results obtained we can therefore say that in more realistic assumptions of market, the switch strategy proposed by the two Authors improves the performance obtained from the model originally formulated.

Keywords: defined contribution pension scheme, switch criterion, accumulation phase, decumulation phase, buffer, mortality risk

1. Introduction

In this work we have made a more thorough analysis of the model suggested by the Authors Arts, Vigna[1] "A switch criterion for defined contribution pension schemes" in order to verify if considering more realistic aspects of the market conditions in which to operate, the model itself might allow the achievement of a superior performance. It has to do with a model that studies the financial risk borne by members of a defined contribution pension fund. In fact in literature there are two different approaches for assessing this type of risk, the theoretical one, aimed at optimizing the risk/yield profile by using dynamic programming techniques, and the empirical one, that examines the most commonly used strategies, trying to find the best alternative among all those possible.
The model suggested by the two Authors closely follows this latter approach and we have developed its analysis in the following points:

- Introduction of the correlation between the financial assets composing the investment portfolio;
- Introduction of new models for the reserves calculation;
- Analysis of the mortality risk for the distribution phase.

The basic strategy is the same, that is we consider two different types of switches, rather than considering a gradual switch of the portfolio to bonds. In fact, once the individual has begun to invest all contributions in equities, at some point that we call time of first switch (SC), he will compose a less risky fund through contributions that, from that very moment on, will be invested in bonds. From the time of first switch, as underlined by the Authors, we will therefore have two funds, the first obtained by investing contributions in equities since the initial time (time of joining the social security scheme) until the time of first switch, and the latter, obtained from SC onwards,by investing the following contributions in bonds until retirement. The period in which the second switch will take place is exactly the time interval between SC and the time of retirement, during which, at the occurrence of certain conditions that we will fully specify below, the equity fund will be converted into bonds. The peculiarity of the strategy suggested by the Authors is precisely to find the best time in which the two switches can take place, in order to maximize the probability of reaching the target fund at retirement. Another remark that must be made is the following, the Authors assume that the period in which the second switch can take place can even be later than the time of retirement, should the retiree decide to take the income drawdown option.

[1] Arts, Vigna, A Switch criterion for defined contribution pension schemes, Working paper 30/03, CeRP.

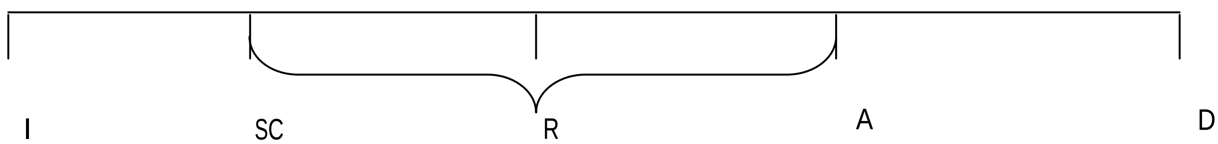
Another innovative aspect of the work suggested by the two Authors was to jointly consider both the accumulation phase and the distribution phase, considering different basic assumptions for the latter, depending on whether there has been a choice between annuitization / drawdown option and depending on whether the second switch has taken place or not. To identify these two periods, as already suggested by the Authors, we will refer to the Monte Carlo simulation, comparing the obtained results with other strategies.

This paper will thus be structured according to the following scheme:

a) test the effectiveness of the model when we make more realistic assumptions on the considered financial assets , such as those of correlation;
b) test the effectiveness of the model itself with correlated activities in the presence of reserves, by providing two new methods for their calculation;
c) identify the strategies that allow to achieve the best results and jointly analyze them during both the accumulation and distribution phase;
d) proceed with the examination of the optimal strategies when considering the risk of mortality within the distribution phase.

2. The Suggested Model

For comparison homogeneity we will use the same symbology and formulations used by the Authors.
For this purpose, we refer to the following payment schedule:



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I Time when the individual joins the pension fund (usually set equal to 0); it is also the moment when the individual starts investing all contributions in equities;

SC Time of First Switch, when the individual stops investing contributions in equities and starts investing them in bonds;

R Time of retirement;

A Time when the purchase of an annuity is compulsory. Under the hypothesis of drawdown option, as in this work, A corresponds to 75 years;

D Time of death of the individual

We must underline that, at the SC time, we actually come to have two funds: the first one is the equity fund, obtained from investing all contributions from I to SC in equities; the latter, that we have starting from SC on, is obtained from investing contributions in bonds. We can place the SF time of second switch between the SC time and the A time, according to what assumed by the Authors. In fact in the introduction we have already mentioned how the second switch SF may take place before or after the R time. More specifically, if SF takes place between SC and R, then the equity fund is fully converted in equities. In such eventuality we will therefore have two financial portfolios fully invested in bonds. After reaching R, the individual leaves the social security scheme and, with the overall settled fund, will purchase a life annuity. If SF does not intervene in this time interval the switch will take place between R and A as the individual takes the income drawdown option and, in this case, instead of converting the equity fund in bonds he will immediately purchase the annuity with the fund settled up at that time. Should the individual reach A without the occurrence of the second switch, then he will have to leave the scheme and compulsorily purchase the fixed annuity with the remaining both equity and stock funds. In this regard, we make the appropriate assumptions on the considered assets. The financial assets are distributed in a log-normal way, in detail, the yield for bonds is e^{μ_t} with $\mu_t \sim N(\mu;\sigma_\mu^2)$, while the yield for equities is e^{λ_t} with $\lambda_t \sim N(\mu;\sigma_\lambda^2)$. The assets are assumed correlated. In order to proceed with the search for the two times SC and SF, the individual, at the time of his joining the pension scheme, must already consider his average rate of actual yield required by the investment, in other words, the Target Fund at Retirement.

As underlined by the Authors, the required yield is linked to the risk aversion by the individual himself and in the work is supposed to be always included between the expected yield $\left(e^{\mu+0,5\sigma_\mu^2}\right)$ on bonds and the expected yield on shares $\left(e^{\lambda+0,5\sigma_\lambda^2}\right)$.We define such expected yield as r* that we calculate as: $r* = \frac{1}{2}(\mu+\lambda)+\frac{1}{8}\left(\sigma_\mu^2+\sigma_\lambda^2+2\rho\sigma_\mu\sigma_\lambda\right)$.

2.1 The Search for the Sc Time

We report in this regard the formulas already underlined by the Authors in their work.

In particular, the SC time is the solution to the following equation:

$$F_I^{TAR} = F_I^{CE} + F_I^{CB} \qquad (1)$$

in which:

$F_I^{TAR} = \sum_{j=0}^{R-1} c(e^{r*})^{(R-j)}$ is the fund at the R time, obtained from investing all contributions at a r*rate;

$F_I^{CE} = \left[\sum_{i=0}^{SC-1} c\left(e^{\lambda+0,5\sigma_\lambda^2}\right)^{(SC-i)}\right]\left(e^{\mu+0,5\sigma_\mu^2}\right)^{(R-SC)}$ is the fund at the R time obtained from investing contributions from I to SC in equities with expected yield $\left(e^{\lambda+0,5\sigma_\lambda^2}\right)$ and then investing this fund for (R-SC) years in bonds with expected yield $\left(e^{\mu+0,5\sigma_\mu^2}\right)$;

$F_I^{CB} = \sum_{i=SC}^{R-1} c\left(e^{\mu+0,5\sigma_\mu^2}\right)^{(R-i)}$ is the fund obtained from investing contributions in bonds for (R-SC) years.

Note that SC can be calculated deterministically once defined the assets yields; moreover is good to point out that the funds described above are calculated by using the expected yields while the actual funds at a generic t time depend of course on the actual realizations of yields over time .

We therefore define the following fund values for a generic t time with $I = 0 \leq t \leq SC$:

$f_t^{CE} = \left(\sum_{i=0}^{t-1} c\prod_{j=i+1}^{t} e^{\lambda_j}\right)$ the value of the fund at t, obtained from investing contributions in equities;

$f_t^{CB} = 0$ the value of the fund at t obtained from investing contributions in bonds (it is equal to 0 because we are before the SC time);

$f_t^{TOT} = f_t^{CE} + f_t^{CB}$ the value of the overall fund settled up to t.

2.2 The Definition of the Switch Criterion for the Equity Fund

According to the analyzed model, the switch of the equity fund can take place every year since the SC time, if this condition occurs:

$$F_{SC}^{TAR} \leq F_{SC}^{CE} + F_{SC}^{CB} \qquad (2)$$

in which the expressions F_{SC}^{TAR} and F_{SC}^{CB} exactly coincide with those described in (1).

We should instead linger over the expression of the fund F_{SC}^{CE} as, once arrived to SC, the actual yields of equities are known and we should therefore refer to them now in order to describe the value of F_{SC}^{CE}.

In particular it will be:

$$F_{SC}^{CE} = \left(\sum_{i=0}^{SC-1} c\prod_{j=i+1}^{SC} e^{\lambda_j}\right)\left(e^{\mu+0,5\sigma_\mu^2}\right)^{(R-SC)} \qquad (3)$$

Let's see now the criterion proposed by the Authors in order to identify the SF time.

If we are at the SC time, given the equality of the funds described above, then the condition of switch becomes simply:

$$\sum_{i=0}^{SC-1} c\prod_{j=i+1}^{SC} e^{\lambda_j} \geq \sum_{i=0}^{SC-1} c\left(e^{\lambda+0,5\sigma_\lambda^2}\right)^{(SC-i)} \qquad (4)$$

that is, if $F_{SC}^{CE} \geq F_I^{CE}$, then the switch takes place if the actual yields have had higher performance than the expected yields.

If this doesn't happen and $F_{SC}^{CE} < F_I^{CE}$, then the switch of the equity fund doesn't take place and the fund remains invested in equities for another year.

In general, it will be true that for any t time such that SC<t<R the switch takes place if this condition occurs:

$$F_t^{CE} + F_{SC,t}^{CB} \geq F_I^{TAR} - F_{t,R}^{CB} \qquad (4a)$$

where:

$F_t^{CE} = f_t^{CE}\left(e^{\mu+0,5\sigma_\mu^2}\right)^{(R-t)}$ is the value of the contributions invested in equities for $SC < t \leq R$ in which $f_t^{CE} = \left(\sum_{i=0}^{SC-1} c \prod_{j=i+1}^{SC} e^{\lambda_j}\right) \prod_{i=SC}^{t} e^{\lambda_i}$ represents the value of the fund with contributions invested in equities if the switch hasn't occurred;

$F_{SC,t}^{CB} = \left(\sum_{i=SC}^{t-1} c \prod_{j=i+1}^{t} e^{\mu_j}\right)\left(e^{\mu+0,5\sigma_{\lambda\mu}^2}\right)^{(R-t)}$ is the value of the fund obtained from investing contributions in bonds from SC to t;

F_I^{TAR} is the target fund at retirement;

$F_{t,R}^{CB} = \sum_{i=t}^{R-1} c\left(e^{\mu+0,5\sigma_\mu^2}\right)^{(R-i)}$ is the value of the fund obtained from investing the future contributions in bonds.

We must point out that if the second switch SF occurs before the t time, then the expression for the fund f_t^{CE} becomes:

$$f_t^{CE} = \left(\sum_{i=0}^{SC-1} c \prod_{j=i+1}^{SC} e^{\lambda_j}\right) \prod_{i=SC+1}^{SF} e^{\lambda_i} \prod_{i=SF+1}^{t} e^{\mu_i}$$

Shouldn't the second switch occur by R, then the individuals will choose the income drawdown option and the switch criterion will also tend to change in a way that we will better define below.

3. Case Study

Let's then start from this early information and test the strategy when the considered assets are correlated and let's see how things change.

Being:

$\mu_t \sim N\left(1{,}5\%; (4\%)^2\right)$ and $\lambda_t \sim N\left(4\%; (9\%)^2\right)$, c=1 and ρ=0,5 x=25, R= 40.

According to this information we will have:

r*=2,9163%, $F_I^{TAR} = F_R^{TAR} = 75{,}92$, $F_I^{CE} = 60{,}84$; $F_I^{CB} = 15{,}08$

Now, by applying the (1), we get SC=27, this means that for the first 26 years contributions are invested in equities and only later in bonds. Once detected the SC time we have provided for detecting the time of the second switch SF. For this purpose we have produced 1000 Monte Carlo simulations concerning the yields of the due assets by applying for every year from SC to R the switch conditions respectively described by (4) and by (4a).
As additional information we have calculated the final fund settled up at R:

$F_R^{TOT} = F_R^{CE} + F_R^{CB}$ where $F_R^{CE} = f_R^{CE}$ and $F_R^{CB} = \sum_{i=SC}^{R} c \prod_{j=i+1}^{R} e^{\mu_j}$

So, as suggested by the two Authors, we have then compared the obtained results with a strategy in which contributions are integrally invested in equities for 40 years.

See the results in the following chart:

	40 years 100% equities	**SC=27**
Mean	117,54	78,44
Standard Deviaton	63,91	44,12
Downside deviation[2]	17,65	7,20
Mean Shortfall[3]	14,78	5,62
P (F_R^{TOT}<F_R^{TAR})	34,20%	45,80%
P (F_R^{TOT}<F_R^{TAR} |SF< 41)	does not apply	28,21%

Table 1: Switch Strategy with SC=27 versus 40 years 100% Equities Strategy

Even in the presence of correlation we can confirm the conclusions reached by the Authors in the previous work, that is:

- The value of the average fund at maturity is much higher for the 100% equities strategy than for the switch strategy ;
- The switch strategy has a higher probability of failing the target, 45.80% compared to the equity strategy, while all other measures of risk range to decidedly lower values;
- For the switch strategy it should then be noted that the probability of failing the target after the intervention of the second switch SF is equal to 28.21%, this means that the switch strategy is not enough almost on 2 cases out of 7, which turns out to be quite a high percentage.

The only thing that we can now repeat as pointed out by the Authors, is that after the SF intervention, the equity risk is eliminated while the risk concerning yield on bonds still remains. In order then to control the increase of the fund over time, the Authors introduced the concept of annual target.

In particular, the definition of its expression is different depending on the fact that we are before or after SC.
For $0 \leq t \leq SC$ we will have:

$$YT_t = \sum_{i=0}^{t-1} c\left(e^{\lambda+0,5\sigma_\lambda^2}\right)^{(t-i)} \qquad (5)$$

while for $SC < t \leq R$ we will have:

$$YT_t = \sum_{i=0}^{SC-1} c\left(e^{\lambda+0,5\sigma_\lambda^2}\right)^{(SC-i)}\left(e^{\mu+0,5\sigma_{\lambda\mu}^2}\right)^{(t-SC)} + \sum_{i=SC}^{t-1} c\left(e^{\mu+0,5\sigma_\mu^2}\right)^{(t-i)} \qquad (6)$$

and we moreover define the following magnitude: $f_t^{TOT} = f_t^{CE} + f_t^{CB}$.

The purpose of this adjustment on the basic model suggested by the two Authors was to study what happens supposing that the SC time is moved forward, in other words analyze what happens if we decide to invest a larger number of years into equities. That means evaluate the probability that the annual target fund (YT_{28},YT_{29},…..,YT_{31}), for which the SC time is fixed in our case to 27, could be reached in the same year.

We report the achieved results:

Year t	**27**	**28**	**29**	**30**	**31**
Average $f_{sc=t}^{TOT}$	61,5	63,2	65,6	67,4	69,6
YT_{SC} at t	48,8	49,6	50,4	51,2	52,1
P (SF=SC|SC= t)	34,7%	40,6%	46,7%	50,2%	53,9%
P(SF≤ t|SC=27)	34,7%	42,3%	49,9%	54,5%	65,4%
P (F_R^{TOT}<F_R^{TAR} |SC= t)	28,2%	25,2%	22,0%	21,0%	18,9%

Table 2: Results Obtained Investing Contributions Extra Years in Equities

2 $\sqrt{\frac{1}{k}\sum_{i=1}^{k}\left(F_R^{TOT_j} - F_R^{TAR}\right)^2}$ where $\left(F_R^{TOT_j} - F_R^{TAR}\right) < 0$ for j=1,2,…,k

3 $\frac{1}{k}\sum_{i=1}^{k}\left(F_R^{TAR} - F_R^{TOT_j}\right)$ where $\left(F_R^{TAR} - F_R^{TOT_j}\right) > 0$ for j=1,2,…,k

What is clear is that the investment in equities for several years leads to a difference between the total fund at t and the annual target at t which is on increasing average. It is also clear that the two probabilities **P (SF=SC|SC= t)** and **P(SF≤ t|SC=27)** are both increasing in time, however, the first always assumes lower values than the latter. In fact, as already restated by the Authors, this is due to the fact that while the probability **P(SF=SC|SC= t)** is tested only at the time t = SC, the other one is tested in each time between SC and t, therefore is the result of the sum of all the switches occurred from 27 to t. It is also worth noting that investing in equities for more years, as it was reasonable to expect, leads to a reduction in the probability of failing the target, as already highlighted by the results obtained by the two Authors.

4. The Introduction of Reserves

A further step taken by the two Authors in their study was to examine the volatility of the target fund at maturity F_R^{TAR}, assuming as a working hypothesis that the annual target YT_t is actually achieved in the subsequent t times, with SC ≤ t ≤ t R. Assuming that the target is precisely enough for example at the beginning of the year 27 is equivalent to affirm the following relationship $F_{27}^{CE} = F_{27}^{TAR} - F_{27}^{CB}$, therefore the final retirement fund will be:

$$YT_{27} \prod_{i=27}^{40} e^{\mu_i} + \sum_{i=27}^{39} c \prod_{j=i+1}^{40} e^{\mu_j}$$

To this end, using the 1000 simulations made before (considering the estimated yields for the last ten years) we have produced the following results:

	27	**28**	**29**	**30**	**31**	**32**	**33**	**34**	**35**	**36**	**37**	**38**	**39**	**40**
SMS	8,2	7,9	7,4	7,1	7,2	7,0	6,5	6,1	5,6	5,0	4,5	3,9	3,1	2,1
DD	10,0	9,6	9,1	8,8	8,7	8,5	8,0	7,5	6,9	6,2	5,6	4,8	3,8	2,6
$P(F_R^{TOT} < F_R^{TAR} \mid f_t^{TOT} = YT_t$	52,5 %	52,7 %	53,5 %	54,0 %	52,7 %	51,2 %	52,5 %	51,4 %	50,9 %	50,3 %	48,8 %	50,1 %	48,9 %	47,2%

Table 3: Results when $f_t^{TOT}=YT_t$

In which we have indicated with:

SMS: simulated mean shortfall from F_I^{TAR};
SDD: simulated downside deviation from F_I^{TAR}.

We basically see two things:

1. As SF increases, the downside deviation and the mean shortfall increase as well;
2. The probability of failing the target through the switch criterion is very high.

In fact, this was a reasonable result to be expected because, as SF increases, the age of retirement is closer and therefore the risk associated with yields on bond has a lower influence on the final fund. Therefore, in light of what produced by our results, the presence of correlation between financial assets makes even more evident the need for reserves, and this makes necessary a change in the switch criterion. As suggested by the Authors, a first way to increase the expected value of the final retirement fund is simply to invest in equities for more years.
In this regard, according to the already conducted analysis, we set SC = 31. In this way we invest contributions in equities up to 30 years, then we make 1000 Monte Carlo simulations and we compare the thus obtained results with the so-called "lifestyle strategy"[4], where it is assumed to gradually disinvest the 10% of equities in bonds during the last 10 years before retirement.

[4] For the definition of lifestyle strategy refer to Arts. Vigna, A Switch criterion for defined contribution pension schemes, Working paper 30/03, CeRP, p. 12.

We report the obtained results:

	40 years 100% equities	**lifestyle strategy**	**SC=27**	**SC=31**
Mean	117,54	113,87	78,44	101,94
Standard Deviaton	63,91	63,14	44,12	52,11
Downside deviation	17,65	24,14	7,20	5,50
Mean Shortfall	14,78	20,61	5,62	4,06
P (F_R^{TOT}<F_R^{TAR})	34,20%	47,00%	45,80%	40,10%
P (F_R^{TOT}<F_R^{TAR} |SF< 41)	does not apply	does not apply	28,21%	18,94%

Table 4: "Lifestyle Strategy" and "100% Equity Strategy" versus Switch Strategy for both SC=27 and SC=31

The results show that by changing SC from 27 to 31, the average fund, while remaining on lower levels than both the equity fund and the fund obtained with the lifestyle strategy, increases from 78.44 to 101.94. Also other measures of risk for the switch strategy are lower, however, if we refer to **P (F_R^{TOT}<F_R^{TAR}),** we see how the one relating to a pure stock investment turns out to be the lowest compared to all the models taken into account, while the one relating to the lifestyle strategy turns out to be the highest of all. It is therefore clear that to simply move forward the time of the first switch is not a sufficient measure to implement the model performance. Another way followed by the Authors to increase the final expected value of the fund was to define again the switch criterion, providing in this case the presence of reserves. More generally, the switch of the equity fund in the presence of a reserve is realized when this condition occurs:

$$F_t^{CE} + F_{SC,t}^{CB} \geq (1 + \text{Buffer})(F_I^{TAR} - F_{t,R}^{CB}) \qquad (7)$$

As definition of the Buffer, the two Authors suggested two different solutions:

1. $B = \frac{SMS_t}{YT_t}$ (8)

2. $B' = \frac{SMS_t}{YT_t\left(e^{\mu+0,5\sigma_\mu^2}\right)^{(R-t)}}$ (8a)

in which is clear that in the second formulation is taken into account the discounted estimated shortfall at a t time. The buffer thus defined clearly depends on the conducted simulations, thus, in order to make the data more homogeneous, we have made a regression of the same on the future t[5] years. We have calculated the results with both the proposed methodologies and we have limited ourselves to report the ones that produce better results by providing an immediate comparison with the "100% equities" strategy which now appears to be preferable in terms of average fund at maturity.

	SC=27	**27 with buffer**	**40 years 100% equities**
Mean	78,44	94,26	117,54
Standard Deviaton	44,12	38,79	63,91
Downside deviation	7,20	4,41	17,65
Mean Shortfall	5,62	3,65	14,78
P (F_R^{TOT}<F_R^{TAR})	45,80%	39,40%	34,20%
P (F_R^{TOT}<F_R^{TAR} |SF< 41)	28,21%	10,75%	does not apply

Table 5: "100% Equity Strategy" versus Switch Strategy for both SC=27 and SC=27 with Buffer

[5] In these two examples, the regression was linear.

	SC=31	31 with buffer	40 years 100% equities
Mean	101,94	110,97	117,54
Standard Deviaton	52,11	48,93	63,91
Downside deviation	5,50	2,99	17,65
Mean Shortfall	4,06	2,71	14,78
P (F_R^{TOT}<F_R^{TAR})	40,10%	37,40%	34,20%
P (F_R^{TOT}<F_R^{TAR} |SF< 41)	18,94%	7,83%	does not apply

Table 6: "100% Equity Strategy" versus Switch Strategy for both SC=31 and SC=31 with Buffer

In both the studied cases, it is clear that the introduction of reserves in the switch strategy produces as immediate result an increase of the average fund at maturity, even if we never succeed in reaching the level of the equity fund. Particularly, in relation to switch strategies, is the "31with buffer" strategy that produces the best results in terms of average fund, with a negative difference of about 5.60% compared to the equity fund, moreover, the introduction of reserves in the switch strategy further reduces all measures of risk compared to the "100% equities" strategy. As for the probability **P (F_R^{TOT}<F_R^{TAR})** the equity strategy turns out to be the least risky. In detail for the switch strategies is evident that the introduction of reserves produces a lowering of this probability and also in this case, is the "31 with buffer" strategy that produces the closest result to that of the "100% equities" strategy (respectively 37.4% and 34.2%). Our contribution, innovative if compared to the work done, was to try to find new ways of reserves calculation, which could allow to obtain better results also and especially with reference to this probability. In this regard we have provided for identifying two new procedures for the buffer calculation. Maintaining the idea of the ratio between SMS and annual target fund, what we have identified is a new method for the ratio numerator calculation.

a) Method n.1

First we define k the total number of simulations in which, for every considered t, occurs the following ratio $\left(F_R^{TAR} - F_R^{TOT_j}\right) > 0$ .In this method we have argued for "macro-results", in the sense that for each future year from SC to R we have calculated, taking into account the results obtained with the k simulations, the following differences:

$$\left(F_R^{TAR} - F_R^{TOT_j}\right)_{min}^{(t)} \quad \text{and} \quad \left(F_R^{TAR} - F_R^{TOT_j}\right)_{Max}^{(t)}$$

For each of the k simulations produced each year $\left(F_R^{TAR} - F_R^{TOT_j}\right)^{(t)}$, we will make a comparison between this magnitude and the Mean Shortfall (MS) previously calculated, moreover we have calculated the following differences $MS - \left(F_R^{TAR} - F_R^{TOT_j}\right)_{min}^{(t)}$ and $\left(F_R^{TAR} - F_R^{TOT_j}\right)_{Max}^{(t)} - MS$.

The achieved results are indicated in the following chart:

	27	28	29	30	31	32	33	34	35	36	37	38	39	40
$(F_R^{TAR}-F_R^{TOTj})_{Max}$	23,84	23,39	24,65	23,96	22,87	21,83	20,95	19,58	20,14	18,29	17,22	13,92	10,42	8,02
$(F_R^{TAR}-F_R^{TOTj})_{min}$	0,019	0,001	0,005	0,039	0,091	0,020	0,023	0,011	0,000	0,005	0,022	0,002	0,011	0,006
MS	8,24	7,85	7,42	7,11	7,22	6,95	6,49	6,05	5,60	4,98	4,50	3,88	3,06	2,10
MS- $(F_R^{TAR}-F_R^{TOTj})_{min}$	8,22	7,85	7,41	7,08	7,12	6,93	6,46	6,04	5,60	4,97	4,47	3,88	3,05	2,10
$(F_R^{TAR}-F_R^{TOTj})_{Max}$-MS	15,60	15,53	17,23	16,85	15,65	14,87	14,46	13,53	14,54	13,31	12,73	10,04	7,35	5,91
MS- $(F_R^{TAR}-F_R^{TOTj})_{min}$	0,23%	0,02%	0,07%	0,55%	1,26%	0,28%	0,35%	0,19%	0,01%	0,10%	0,49%	0,04%	0,35%	0,30%
$(F_R^{TAR}-F_R^{TOTj})_{Max}$-MS	47,19%	49,45%	56,96%	57,77%	53,90%	53,26%	55,14%	55,25%	61,47%	62,60%	64,67%	61,30%	58,31%	64,46%

Table 7: Calculations for new Reserves with Method n.1

What emerges is:

$$MS \cong MS - \left(F_R^{TAR} - F_R^{TOT_j}\right)_{min}^{(t)}$$

and as it was reasonable to expect slightly higher. In terms of produced results we can see that on average for each year their difference is about 0.3%.

See also the following chart on support:

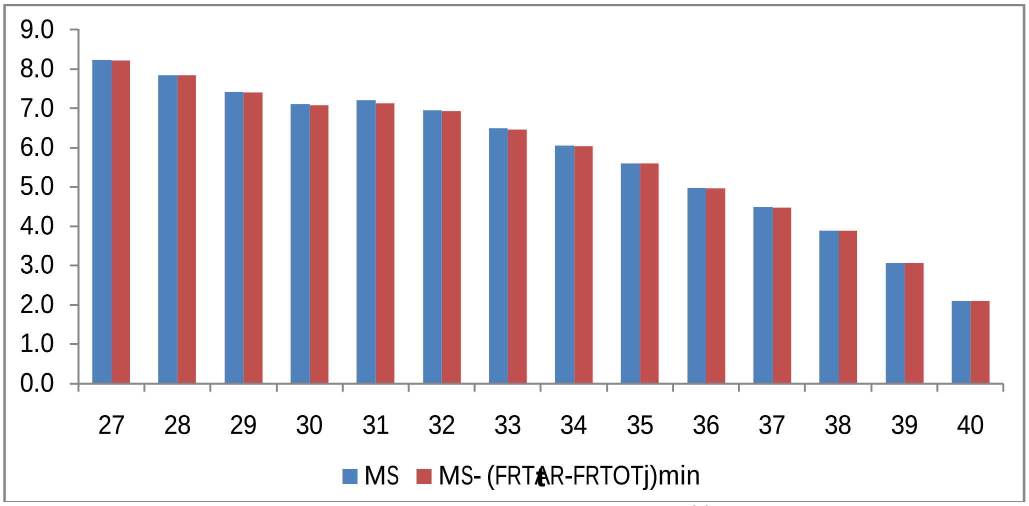

Chart 1: MS $-\left(\mathbf{F_R^{TAR}} - \mathbf{F_R^{TOT_j}}\right)_{\mathbf{min}}^{(\mathbf{t})}$

On the contrary, the difference $\left(F_R^{TAR} - F_R^{TOT_j}\right)_{Max}^{(t)} - MS$ takes definitely higher values (about the 50,3%) as shown by the following chart:

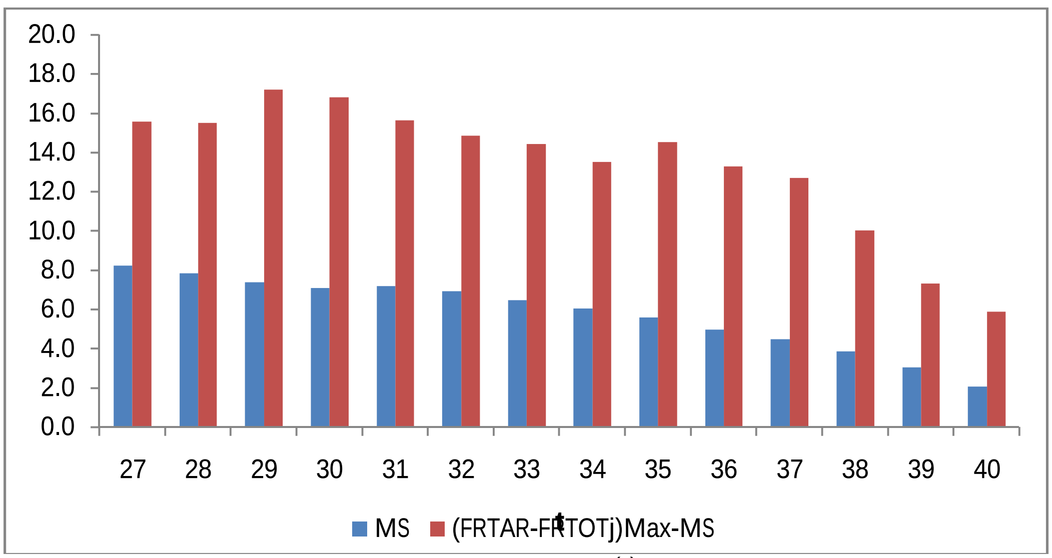

Chart 2: $\left(\mathbf{F_R^{TAR}} - \mathbf{F_R^{TOT_j}}\right)_{\mathbf{Max}}^{(\mathbf{t})} - \mathbf{MS}$

What this analysis allows us to say, is that there is a considerable discrepancy between the two differences and the amount of the MS numerator chosen for the buffer calculation. In particular, while for the difference $MS - \left(F_R^{TAR} - F_R^{TOT_j}\right)_{min}^{(t)}$ the MS chosen as element of allowance is always higher than the worst case scenario in which we could find ourselves in each future year, $\left(F_R^{TAR} - F_R^{TOT_j}\right)_{min}^{(t)}$, for the maximum difference between the same magnitudes we find that the item chosen as buffer numerator turns out to be largely insufficient. Therefore, in the course of this work we tried to find an alternative way in order to integrate this discrepancy. In particular what we acted on is the Buffer numerator, in order to get a more suitable reserve amount. Such new numerator will be denoted by N.

In detail we reasoned as follows

- If $\left(F_R^{TAR} - F_R^{TOT_j}\right)^{(t)} > MS$ then what we put aside is directly the difference $\left(F_R^{TAR} - F_R^{TOT_j}\right)^{(t)}$ always occurring this condition:

$$\left(F_R^{TAR} - F_R^{TOT_j}\right)^{(t)} \leq \left(F_R^{TAR} - F_R^{TOT_j}\right)_{Max}^{(t)}$$

- If $\left(F_R^{TAR} - F_R^{TOT_j}\right)^{(t)} < MS$ then we put aside the magnitude $MS - \left(F_R^{TAR} - F_R^{TOT_j}\right)_{min}^{(t)}$ being always: $MS > MS - \left(F_R^{TAR} - F_R^{TOT_j}\right)_{min}^{(t)}$

With this second method, we will have for each future year, j new estimated $MS_j^{(t)}$ and at this point the new Buffer numerator with this second approach is:

$$N = \sum_{j=1}^{K} \frac{MS_j^{(t)}}{j}$$

with k defined as above.

Therefore as before we define the following buffers

$$B^{(1)'} = \frac{N}{YT_t} \qquad (9)$$

that is:

$$B_t^{(1)''} = \frac{N}{YT_t \left(e^{\mu + 0{,}5\sigma_\mu^2}\right)^{(R-t)}} \qquad (9a)$$

Applying the methodology described above we obtained the following results.

					Method n. 1			
t	B	B regressed	B’ discounted	B’ discounted and regressed	$B^{(1)'}$	$B^{(1)'}$ regressed	$B^{(1)''}$ discounted	$B^{(1)''}$ discounted and regressed[6]
0	0,169	0,167	0,099	0,105	0,217	**0,215**	0,127	**0,122**
1	0,155	0,157	0,095	0,100	0,201	**0,202**	0,122	**0,122**
2	0,142	0,147	0,090	0,095	0,183	**0,189**	0,116	**0,120**
3	0,131	0,136	0,087	0,090	0,169	**0,176**	0,112	**0,117**
4	0,129	0,126	0,089	0,085	0,164	**0,163**	0,113	**0,114**
5	0,120	0,116	0,086	0,080	0,154	**0,149**	0,110	**0,109**
6	0,108	0,106	0,081	0,075	0,140	**0,136**	0,105	**0,104**
7	0,098	0,095	0,076	0,070	0,127	**0,123**	0,099	**0,097**
8	0,088	0,085	0,071	0,065	0,114	**0,110**	0,093	**0,089**
9	0,076	0,075	0,064	0,060	0,099	**0,097**	0,084	**0,081**
10	0,066	0,064	0,058	0,056	0,086	**0,083**	0,076	**0,071**
11	0,055	0,054	0,051	0,051	0,071	**0,070**	0,066	**0,061**
12	0,042	0,044	0,041	0,046	0,055	**0,057**	0,053	**0,049**
13	0,028	0,033	0,028	0,041	0,037	**0,044**	0,037	**0,036**

Table 8: Results for new Reserves with Method n.1, SC=27

[6] The regression in this case was obtained from a quadratic rather than linear ratio.

					Method 1			
t	B	B regressed	B' discounted	B' discounted and regressed	$B^{(1)'}$	$B^{(1)'}$ regressed	$B^{(1)''}$ discounted	$B^{(1)''}$ discounted and regressed
0	0,129	0,131	0,089	0,094	0,164	**0,168**	0,113	**0,121**
1	0,120	0,120	0,086	0,087	0,154	**0,154**	0,110	**0,112**
2	0,108	0,109	0,081	0,081	0,140	**0,140**	0,105	**0,104**
3	0,098	0,098	0,076	0,074	0,127	**0,126**	0,099	**0,096**
4	0,088	0,087	0,071	0,068	0,114	**0,112**	0,093	**0,087**
5	0,076	0,076	0,064	0,061	0,099	**0,098**	0,084	**0,079**
6	0,066	0,065	0,058	0,055	0,086	**0,084**	0,076	**0,071**
7	0,055	0,054	0,051	0,048	0,071	**0,070**	0,066	**0,063**
8	0,042	0,043	0,041	0,042	0,055	**0,056**	0,053	**0,054**
9	0,028	0,032	0,028	0,035	0,037	**0,042**	0,037	**0,046**

Table 9: Results for New Reserves with Method n.1, SC=31

b) Method n.2

In this second method, we reasoned by "micro-results" choosing to start from a different database, that is, for each future year and for each of the k simulations we built the chart of differences $\left(F_R^{TAR} - F_R^{TOTj}\right) - MS$[7] moreover, with reference to this chart, we calculated the following magnitudes:

$$\left[\left(F_R^{TAR} - F_R^{TOTj}\right) - MS\right]_{min}^{(t)} ; \left[\left(F_R^{TAR} - F_R^{TOTj}\right) - MS\right]_{Max}^{(t)} ; \left[\left(F_R^{TAR} - F_R^{TOTj}\right) - MS\right]_{Max}^{(t)} - MS$$

obtaining the following results:

	27	**28**	**29**	**30**	**31**	**32**	**33**	**34**	**35**	**36**	**37**	**38**	**39**	**40**
$[(F_R^{TAR}-F_R^{TOTj})-MS]^{(t)}_{min}$	8,22	7,85	7,41	7,08	7,12	6,93	6,46	6,04	5,60	4,97	4,47	3,88	3,05	2,10
$[(F_R^{TAR}-F_R^{TOTj})-MS]^{(t)}_{Max}]$	15,60	15,53	17,23	16,85	15,65	14,87	14,46	13,53	14,54	13,31	12,73	10,04	7,35	5,91
MS	8,24	7,85	7,42	7,11	7,22	6,95	6,49	6,05	5,60	4,98	4,50	3,88	3,06	2,10
$[(F_R^{TAR}-F_R^{TOTj})-MS]^{(t)}_{Max}]$-MS	7,36	7,68	9,81	9,73	8,44	7,92	7,97	7,47	8,94	8,33	8,23	6,15	4,29	3,81
$[(F_R^{TAR}-F_R^{TOTj})\_MS]^{(t)}_{min}$-MS	0,23%	0,02%	0,07%	0,55%	1,26%	0,28%	0,35%	0,19%	0,01%	0,10%	0,49%	0,04%	0,35%	0,30%
$[(F_R^{TAR}-F_R^{TOTj})-MS]^{(t)}_{Max}$-MS	47,19 %	49,45 %	56,96%	57,77%	53,90%	53,26%	55,14%	55,25%	61,47%	62,60%	64,67%	61,30%	58,31%	64,46 %

Table 10: Calculations for New Reserves with Method n.2

Emerges what follows:

$$MS \cong \left[\left(F_R^{TAR} - F_R^{TOTj}\right) - MS\right]_{min}^{(t)}$$

and as it was reasonable to expect, slightly higher. In terms of produced results we can see that on average for each year their difference is about 0,3%.

[7] These differences are considered in their absolute value

See also the following chart on support:

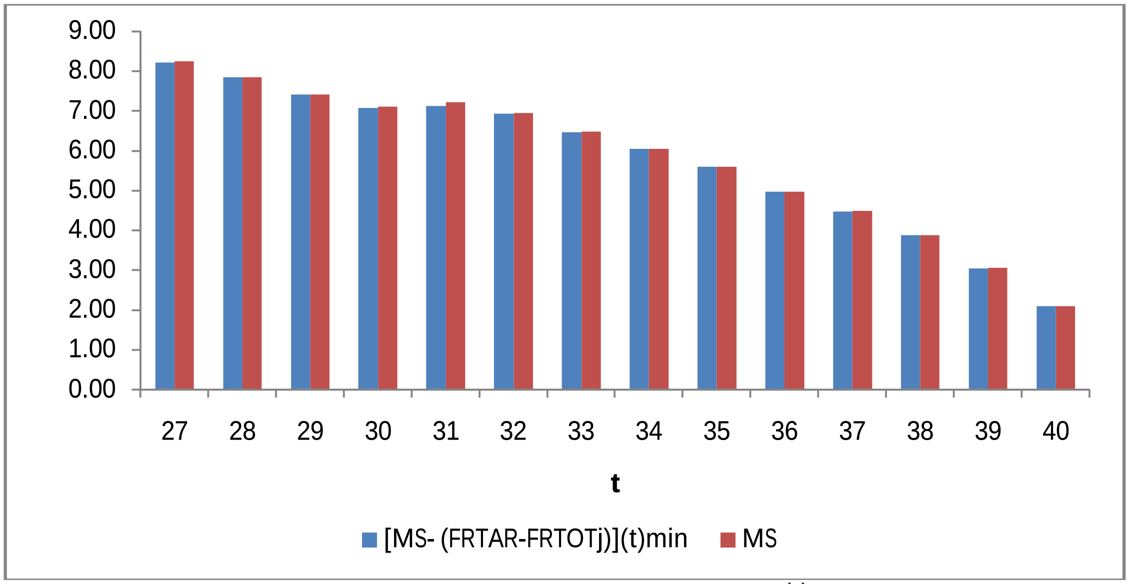

Chart 3: MS e $\left[\left(\mathbf{F_R^{TAR}} - \mathbf{F_R^{TOTj}}\right) - \mathbf{MS}\right]_{\mathbf{min}}^{(\mathbf{t})}$

On the contrary the difference $\left[\left(F_R^{TAR} - F_R^{TOTj}\right) - MS\right]_{Max}^{(t)}$ takes definitely higher values (about the 57,3%) as highlighted in the following chart:

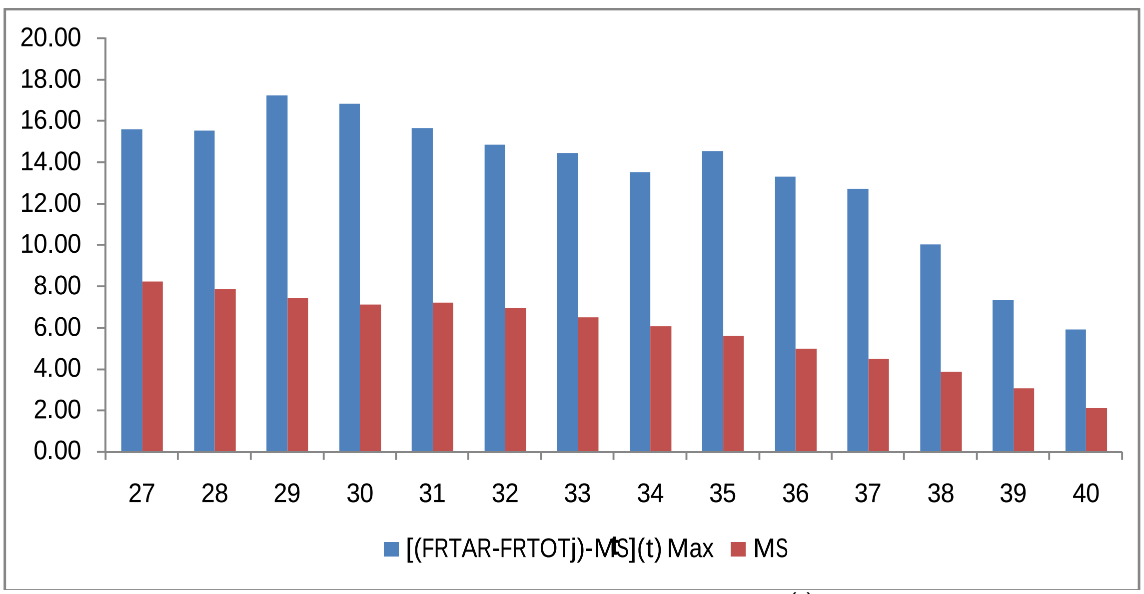

Chart 3: MS e $\left[\left(\mathbf{F_R^{TAR}} - \mathbf{F_R^{TOTj}}\right) - \mathbf{MS}\right]_{\mathbf{MAx}}^{(\mathbf{t})}$

Also this analysis allows us to reach the same conclusions outlined for the Method 1: there is a considerable discrepancy between the two differences and the amount of the MS numerator chosen for the buffer calculation.

In particular, while for the difference $\left[\left(F_R^{TAR} - F_R^{TOTj}\right) - MS\right]_{min}^{(t)}$ the MS chosen as element of allowance is always higher than the worst case scenario in which we could find ourselves in each future year $\left[\left(F_R^{TAR} - F_R^{TOTj}\right) - MS\right]_{min}^{(t)}$, for the maximum difference between the same magnitudes we find that the item chosen as buffer numerator turns out to be largely insufficient.

Therefore also in this case, in order to reduce such discrepancy, we tried to modify again the Buffer numerator, so as to be able to get a more suitable reserve amount.

This new numerator will be denoted by N.

In detail, we reasoned as follows:

- If $\left[\left(F_R^{TAR} - F_R^{TOT_j}\right) - MS\right]^{(t)} > MS$ then what we put aside is directly the difference $\left[\left(F_R^{TAR} - F_R^{TOT_j}\right) - MS\right]^{(t)} - MS$ always occurring this condition:

$$\left[\left(F_R^{TAR} - F_R^{TOT_j}\right) - MS\right]^{(t)} \leq \left[\left(F_R^{TAR} - F_R^{TOT_j}\right) - MS\right]_{Max}^{(t)}$$

- If $\left[\left(F_R^{TAR} - F_R^{TOT_j}\right) - MS\right]^{(t)} < MS$ then we put aside the magnitude

$\left[\left(F_R^{TAR} - F_R^{TOT_j}\right) - MS\right]_{min}^{(t)}$ being always:

$$MS > \left[\left(F_R^{TAR} - F_R^{TOT_j}\right) - MS\right]_{min}^{(t)}$$

Also with this second method we will have for each future year, j new estimated $MS_j^{(t)}$ and at this point the new Buffer numerator with this second approach is:

$$N' = \sum_{j=1}^{K} \frac{MS_j'^{(t)}}{j}$$

with k defined as above.

As before we therefore describe the following buffers:

$$B^{(2)'} = \frac{N'}{YT_t} \qquad (10)$$

that is

$$B_t^{(2)''} = \frac{N'}{YT_t\left(e^{\mu+0,5\sigma_\mu^2}\right)^{(R-t)}} \qquad (10a)$$

Applying the methodology described above we obtained the following results.

					Method n.2			
t	B	B regressed	B' discounted	B' discounted and regressed	$B^{(2)'}$	$B^{(2)'}$ regressed	$B^{(2)''}$ discounted	$B^{(2)''}$ discounted and regressed[8]
0	0,169	0,167	0,099	0,105	0,174	**0,173**	0,102	**0,094**
1	0,155	0,157	0,095	0,100	0,161	**0,163**	0,098	**0,095**
2	0,142	0,147	0,090	0,095	0,148	**0,152**	0,094	**0,094**
3	0,131	0,136	0,087	0,090	0,136	**0,142**	0,090	**0,093**
4	0,129	0,126	0,089	0,085	0,132	**0,131**	0,091	**0,091**
5	0,120	0,116	0,086	0,080	0,124	**0,120**	0,089	**0,088**
6	0,108	0,106	0,081	0,075	0,113	**0,110**	0,085	**0,084**
7	0,098	0,095	0,076	0,070	0,102	**0,099**	0,080	**0,080**
8	0,088	0,085	0,071	0,065	0,092	**0,089**	0,075	**0,074**
9	0,076	0,075	0,064	0,060	0,080	**0,078**	0,068	**0,068**
10	0,066	0,064	0,058	0,056	0,069	**0,067**	0,061	**0,061**
11	0,055	0,054	0,051	0,051	0,058	**0,057**	0,054	**0,054**
12	0,042	0,044	0,041	0,046	0,044	**0,046**	0,043	**0,045**
13	0,028	0,033	0,028	0,041	0,030	**0,036**	0,030	**0,036**

Table 11: Results for New Reserves with Method n.2, SC=27

[8] Also in this case, the regression was obtained by quadratic rather than linear ratio.

					Method n.2			
t	B	B' regressed	B discounted	B' discounted and regressed	$B^{(2)'}$	$B^{(2)'}$ regressed	$B^{(2)''}$ discounted	$B^{(2)''}$ discounted and regressed[9]
0	0,129	0,131	0,089	0,094	0,132	**0,135**	0,091	**0,096**
1	0,120	0,120	0,086	0,087	0,124	**0,124**	0,089	**0,089**
2	0,108	0,109	0,081	0,081	0,113	**0,113**	0,085	**0,083**
3	0,098	0,098	0,076	0,074	0,102	**0,101**	0,080	**0,077**
4	0,088	0,087	0,071	0,068	0,092	**0,090**	0,075	**0,070**
5	0,076	0,076	0,064	0,061	0,080	**0,079**	0,068	**0,064**
6	0,066	0,065	0,058	0,055	0,069	**0,068**	0,061	**0,058**
7	0,055	0,054	0,051	0,048	0,058	**0,056**	0,054	**0,051**
8	0,042	0,043	0,041	0,042	0,044	**0,045**	0,043	**0,045**
9	0,028	0,032	0,028	0,035	0,030	**0,034**	0,030	**0,039**

Table 12: Results for New Reserves with Method n.1, SC=31

As we can see from the obtained results we can say that while the new method n. 2 of reserves calculation leads to however intermediate buffer values compared to those calculated with the methodology suggested by the Authors, the method n.1leads to however higher buffer values. At this point, taking the buffer values calculated with the first and second method, we tested again the suggested switch strategies, however limiting ourselves to report the results of those which are more significant to us in terms of average fund at maturity and in terms of probability of failing the target.

	SC=27	**27 with buffer**	**27 with buffer (Method n.1)**
Mean	78,44	94,26	98,91
Standard Deviaton	44,12	38,79	37,02
Downside deviation	7,20	4,41	7,15
Mean Shortfall	5,62	3,65	6,87
P (F_R^{TOT}<F_R^{TAR})	45,80%	39,40%	38,90%
P (F_R^{TOT}<F_R^{TAR} |SF< 41)	28,21%	10,75%	7,84%

Table 13: Switch Strategy with SC=27 versus Switch Strategy SC=27 with Buffer and Switch Strategy SC=27 with Buffer (Method n.1)

The results show that the model that provides the best results in terms of performance is the one obtained by applying the methodology of reserves calculation introduced by us, which produces a higher value of the average fund at maturity with smaller risk measures. In any case we must underline that, although we succeed in increasing the amount of the average fund at maturity, we are however never able to match the performance of the equity strategy. Our method of reserves calculation, although reducing the probability of failing the target with respect to the compared cases, however again turns out to be more risky than the equity strategy (38.90% vs. 34.2%). Now we examine the switch strategy with SC = 31. Again in relation to the different methodologies of reserves calculation, we report the results that we consider most significant.

	SC=31	**31 with buffer**	**31 with buffer (Method n.1)**
Mean	101,94	110,97	114,91
Standard Deviaton	52,11	48,93	47,40
Downside deviation	5,50	2,99	1,86
Mean Shortfall	4,06	2,71	1,51
P (F_R^{TOT}<F_R^{TAR})	40,10%	37,40%	33,14%
P (F_R^{TOT}<F_R^{TAR} |SF< 41)	18,94%	7,83%	5,05%

Table 14: Switch Strategy with SC=31 versus Switch Strategy SC=31 with Buffer and Switch Strategy SC=31 with Buffer (Method n.1)

[9] Also in this case, the regression was obtained by quadratic rather than linear ratio.

Even with SC = 31 the best results are obtained by using our new method of reserves calculation. Unlike the two other compared cases, in fact, we obtain a higher average fund with lower risk measures and lower probabilities to fail the target. Moreover, regarding this last statement, the results show that our new methodology of reserves calculation is able to get even better results than the equity strategy (respectively 33.14% and 34.2%). However we must underline something else. Although the introduction of reserves allows us to get higher amounts for the average fund at maturity, we never succeed in reaching the amount obtained with the equity strategy (117,54) even though our method of reserves calculation produces the closest result to this amount (114,91) with a difference in percentage terms of about the 2.3%.

5. The Distribution Phase

Once reached the R time of retirement, the scheme members are left with the following opportunity. Should the second switch SF occur between SC and R, the individuals will have to leave the scheme at their retirement time and, with the overall settled fund, fully invested in bonds, will purchase the pension yield, whose annual amount will be[10]:

$$\text{Pä} = \left(\frac{F_R^{TOT}}{\ddot{a}_{x+R}}\right)$$

Note that the annual pension may be higher or lower than the target pension, depending on the occurrence of $F_R^{TOT} \geq F_R^{TAR}$ that is $F_R^{TOT} \leq F_R^{TAR}$. Shouldn't the second switch SF occur between SC and R, the two Authors insert the possibility of the income drawdown option without bequest, namely the possibility for the individual to keep remaining inside the fund even in periods subsequent to R, while still having the possibility to annually withdraw a R pension rate whose amount is:

$$P = \left(\frac{F_R^{TAR}}{\ddot{a}_{x+R}}\right)$$

However, should the death of the individual occur between R and R +10, the possible accumulated fund will not be distributed to the surviving family members but to the other scheme members. In this case, through the drawdown option, the individual may benefit from additional yields from funds invested in the respective assets for periods longer than R, moreover, should occur the condition for the second switch, he will not convert the equity fund in bonds anymore, but will immediately provide for purchasing a life annuity.

The formula found by the two Authors for the switch criterion after R is the following[11]:

$$F_t^{TOT} = \left[\left(f_t^{PE} + \min\left(f_t^{PB} - P; 0\right)\right)e^{\lambda_t} + \max\left(f_t^{PB} - P; 0\right)e^{\mu_t}\right]\left(1 + \frac{q_t}{p_t}\right) = \text{P}\ddot{a}_{x+t} = F_t^{TAR} \quad (11)$$

in which is clear that the P pension amount is first withdrawn from the bond fund and, only when this is insufficient, from the equity fund. As in the work of the two Authors, we also establish the upper limit of 75 years over which the drawdown option is no longer allowed and the individual must necessarily purchase the pension yield with the remaining funds. What we did was to compare the different strategies with and without reserves in the accumulation and distribution phase, making a comparison in terms of obtained results with the "100% equities" strategy which turns out to be the one with better performance in terms of average fund at maturity.

It should be underlined that, as starting point suggested by the two Authors, we do not make 1000 simulations again, but we go forward only with those that have not achieved the target at R.

[10] For the definition of $\ddot{a}_{x+R}$ see Arts Vigna, A Switch criterion for defined contribution pension schemes, Working paper 30/03, CeRP, p. 16.

[11] $\left(1 + \frac{q_t}{p_t}\right)$ bonus factor for pooling (see Blake et al, 2001).

	SC=27	SC=27 with buffer	SC=27 with buffer (Method n.1)	100% equities
P (F_R^{TOT}<F_R^{TAR})	45,80%	39,40%	38,90%	34,20%
P (F_R^{TOT}<F_R^{TAR} |SF< 41)	28,21%	10,75%	7,84%	does not apply
P(No Drawdown & F_R^{TOT} < F_R^{TAR})	21,30%	7,30%	5,20%	does not apply
E(F_R^{TOT}|(No Drawdown & F_R^{TOT} < F_R^{TAR})	70,30	72,27	73,05	does not apply
P(Drawdown & F_R^{TOT} < F_R^{TAR})	24,5%	32,1%	33,7%	34,2%
E(F_R^{TOT}|(Drawdown & F_R^{TOT} < F_R^{TAR})	58,5	61,6	63,2	61,1
P(Switch between R e R+10)	8,7%	15,4%	16,7%	15,3%
Average year switch after R	5,2	4,0	3,5	4,5
P (0<F_{R+10}^{TOT}<F_{R+10}^{TAR} & F_t^{TOT}≥0; R<t<R+10)	12,0%	13,5%	14,6%	15,0%
E(F_{R+10}^{TOT}|F_{R+10}^{TOT} < F_{R+10}^{TAR})	18,2	20,3	20,1	20,1
P(Ruin)	3,8%	3,2%	2,4%	3,9%
Average year of ruin after R	9,3	9,0	9,0	9,3

Table 15: The Distribution Phase for Switch Strategy SC=27

Let's comment the achieved results, currently limiting ourselves for now to those above the marked line.
The expression **P(No Drawdown & F_R^{TOT} < F_R^{TAR})** represents the probability not to take the income drawdown option when SF has occurred and the target maturity fund has never been reached. In other words, this is the situation where the individual, who leaves the fund purchasing a life annuity, finds himself at R, though obtaining a lower pension amount than hoped. It is clear from the results that this probability decreases when we introduce reserves in the switch strategy, with an average fund that in this eventuality tends instead to increase and turns out to be quite high if we consider that the Target fund at R is equal to 75,92 . The best results are associated with the Method N. 1 of reserves calculation (73,5 and 5,20%). Then, summing the probability of taking the drawdown option with the probability of not taking this option, we exactly get **P (F_R^{TOT}<F_R^{TAR})** which is the probability of failing the retirement target. We have already remarked how this probability tends to decrease when we introduce reserves in the strategy and once again we reach the best result with the method suggested by us for the reserves calculation (38,94%), although we have had occasion to underline that, despite the occurred improvement, the equity strategy is the one that scores best. Let's now consider the achieved results when we consider the income drawdown and therefore the outcome of the switch strategy in the actual distribution phase. All of these are shown below the marked line.

The expression **P(Drawdown & F_R^{TOT} < F_R^{TAR})** represents the probability of start taking the drawdown when the target fund has never been reached. We see how this probability increases as we insert reserves inside the switch model, as well as increases the average fund available to the individual at the beginning of the drawdown itself.
It should be noted that the best result in terms of average fund, when compared with that of the "100% equities" strategy, can just be obtained with the new reserves calculation model introduced by us (63,2). The probability for the switch to occur between R and R+10 increases with the introduction of reserves and assumes the highest level with our new method (Method n. 1), even placing itself on higher values than those of the "100 % equities" strategy. If on one hand the introduction of reserves increases the probability for the second switch to take place between R and R +10, on the other hand their introduction progressively reduces the average number of years in which this second switch occurs and we can note that the Method N. 1 of reserves calculation is the one resulting in the lowest value of all (on average 3,5 years needed to make the switch between R and R +10) The probability of having a number of paths that reaches R +10 without switch and without ruin increases for the switch strategies when reserves are introduced, placing with both calculation methods at a value however lower than that of the equity strategy.

The average fund in this hypothesis turns out to be increasing when we introduce reserves with values quite similar to each other and compared to the equity strategy. Let's finally analyze the probability of ruin. This decreases in the switch model with the introduction of reserves, however, placing itself on lower values compared to the equity strategy, on the other hand, however, we note that with the introduction of reserves becomes slightly lower the average number of years necessary in order for the ruin to occur.

We should note then that:

1) Adding the following probabilities, P ($0<F_{R+10}^{TOT}<F_{R+10}^{TAR}$ &$F_t^{TOT}\geq 0$; R<t<R+10), P(Ruin), and P(Switch between R e R+10) we exactly get P(Drawdown & $F_R^{TOT} < F_R^{TAR}$);
2) Adding P(No Drawdown & $F_R^{TOT} < F_R^{TAR}$), P ($0<F_{R+10}^{TOT}<F_{R+10}^{TAR}$ &$F_t^{TOT}\geq 0$; R<t<R+10), P(Ruin), we get the overall probability (jointly considering accumulation and distribution phase) of not reaching the target for the switch strategy

We report such probabilities in the following chart:

	SC=27	**SC=27 with buffer**	**SC=27 with buffer (Method n.1)**	**100% equities**
Total probability of failing the Target	37,1%	24,0%	22,2%	18,9%

Table 16: Total Probability of not Reaching the Target for the Switch Strategy SC=27

The results clearly show that the introduction of reserves as it was reasonable to expect, significantly lowers such probability; we obtain the best result by applying our calculation method, even if we are never able to reach the values of the equity strategy that, in the distribution phase turns out to be the least risky. Let's now compare the results of the switch strategy with and without reserves when SC = 31 with those of the equity strategy. This switch strategy had actually allowed to obtain performance however similar to the purely equity strategy during the accumulation phase, rather, in some cases, even higher, so let's see what happens when we consider the distribution phase.

	SC=31	**SC=31 with buffer**	**SC=31 with buffer (Method n.1)**	**100% equities**
P ($F_R^{TOT}<F_R^{TAR}$)	40,10%	37,40%	33,14%	34,20%
P ($F_R^{TOT}<F_R^{TAR}$ |SF< 41)	18,94%	7,83%	5,05%	does not apply
P(No Drawdown & $F_R^{TOT} < F_R^{TAR}$)	14,00%	4,90%	3,30%	does not apply
E(F_R^{TOT}|(No Drawdown & $F_R^{TOT} < F_R^{TAR}$)	71,86	73,21	74,41	does not apply
P(Drawdown & $F_R^{TOT} < F_R^{TAR}$)	26,1%	32,5%	31,3%	34,2%
E(F_R^{TOT}|(Drawdown & $F_R^{TOT} < F_R^{TAR}$)	59,7	60,9	61,4	61,1
P(Switch between R e R+10)	10,2%	14,0%	14,2%	15,3%
Average year switch after R	4,4	4,2	4,5	4,5
P ($0<F_{R+10}^{TOT}<F_{R+10}^{TAR}$ & $F_t^{TOT}\geq 0$; R<t<R+10)	13,0%	15,1%	14,1%	15,0%
E(F_{R+10}^{TOT}|$F_{R+10}^{TOT} < F_{R+10}^{TAR}$)	18,8	18,4	20,2	20,1
P(Ruin)	2,9%	3,4%	3,0%	3,9%
Average year of ruin after R	9,1	9,1	9,3	9,3

Table 17: The Distribution Phase for Switch Strategy SC=31

Let's now comment the obtained results, always starting from those placed above the marked line.

Even with SC = 31 the probability **P(No Drawdown & $F_R^{TOT} < F_R^{TAR}$)** decreases when we introduce reserves in the switch strategy, with an average fund that in this eventuality tends instead to increase and turns out to be quite high if we think that the target fund at R is 75,92. You should note that for both cases we obtain the best results by using the reserves calculation method suggested by us (3,30% and 74,41). Then, summing the probabilities of taking the drawdown option and the probability of not taking this option, we exactly get **P ($F_R^{TOT}<F_R^{TAR}$)** which is the probability of failing the retirement target. We have already underlined how such probability tends to decrease when we introduce reserves in the strategy, obtaining the best value by the reserves calculation method introduced by us, that also allows us to reduce the risk compared to the equity strategy (33,14% against 34,2%).

It should be remembered, as already seen before, that in this case, however, we cannot reach the value of the average fund of the equity strategy although for a little difference.

Let's now see the results obtained when we consider the income drawdown and therefore the outcome of the switch strategy in the actual distribution phase. All of these are shown below the marked line. The probability **P(Drawdown & $\mathbf{F_R^{TOT} < F_R^{TAR}}$**) increases when we insert reserves in the switch model, as well as increasing is the average fund available to the individual at the beginning of the drawdown. The best results compared to the equity strategy are obtained by the reserves calculation method introduced by us. The probability for the switch to occur between R e R+10 increases with the introduction of reserves, even if we cannot reach the values of the equity strategy. If instead we analyze the average number of years necessary for the switch to occur between R and R +10, we see that they are quite similar if we consider the strategies SC = 31, SC = 31 with buffer (Method N. 1) and the equity strategy, while this value turns out to be lower in the strategy SC = 31 with buffer. The probability of having a number of paths that reaches R +10 without switch and without ruin increases for the switch strategies when we introduce reserves, even if the strategy SC = 31 with buffer (Method n. 1) produces the lowest result of all (14,1%) with an amount of the average fund that in this case is higher compared to all other examined strategies. Let's finally analyze the probability of ruin. This increases the switch model with the introduction of reserves, even if the strategy SC = 31 with buffer (Method n.1) produces the lowest result of all (3%) with an average number of years for bankruptcy perfectly analogous to the equity strategy.

Also in this case we calculate the overall probability of failing the target and we report the results in the table below:

	SC=31	**SC=31 with buffer**	**SC=31 with buffer (METHOD N.1)**	**100% equities**
Total probability of failing the Target	29,9%	23,4%	19,1%	18,9%

Table 18: Total Probability of not Reaching the Target for the Switch Strategy SC=31

What we note is that the strategy SC = 31 with buffer (Method N. 1) has an overall probability of failing the target much closer to the equity probability (with a difference between the two of about 0,8%) that, jointly considering also the distribution phase, continues to be the least risky (18.9%). In light of the obtained results we can therefore say that with realistic market hypothesis, the switch strategy suggested by the Authors Arts and Vigna allows to get better performance than that highlighted in the original work.

6. Distribution Phase and Mortality Risk

As additional element of novelty compared to the previous work, we have examined what happens to the suggested model when we introduce in the distribution phase the income drawdown option with bequest payable to the member's estate. Referring for this to what proposed by Blake et al. in their work[12], we have always hypothesized the possibility for the individual to withdraw the pension amount P (above defined) between R and R +10, yet, should the individual die before the age of 75 years (in which the purchase of the annuity is compulsory) the residual fund is paid as a bequest to the plan member's estate. In such eventuality it is therefore necessary to slightly modify the switch criterion after the R retirement.

In particular, the second switch will take place after R, in case it occurs what follows[13]:

$$F_t^{TOT} = \left[\left(f_t^{PE} + \min\left(f_t^{PB} - P; 0\right)\right) e^{\lambda_t} + \max\left(f_t^{PB} - P; 0\right) e^{\mu_t}\right] \frac{e^{\mu + 0,5\sigma_\mu^2}}{p_{x+t}} = P\ddot{a}_{x+t} = F_t^{TAR} \quad (12)$$

In light of this new criterion we have reviewed the switch strategies proposed by us both for SC=27 and SC=31 suggesting the usual comparison with the "100% equities" strategy.

[12] Pensionmetrics 2: Stochastic Pension Plan Design During the Distribution Phase, David Blake, Andrew J.G. Cairns and Kevin Dowd, August 2002, The PENSIONS INSTITUTE Birkbeck College, University of London, 7-15 Gresse St. London W1T 1LL, UK

[13] Where $\frac{e^{\mu + 0,5\sigma_\mu^2}}{p_{x+t}}$ represents the factor introduced by Blake in pension schemes with bequest payable to the member' estate.

We start with the SC=27 strategy.

	SC=27	**SC=27 with buffer**	**SC=27 with buffer (Method n.1)**	**100% equities**
P (F_R^{TOT}<F_R^{TAR})	45,80%	39,40%	38,90%	34,20%
P (F_R^{TOT}<F_R^{TAR} |SF< 41)	28,21%	10,75%	7,84%	does not apply
P(No Drawdown & F_R^{TOT} < F_R^{TAR})	21,30%	7,30%	5,20%	does not apply
E(F_R^{TOT}|(No Drawdown & F_R^{TOT} < F_R^{TAR})	70,30	72,27	73,05	does not apply
P(Drawdown & F_R^{TOT} < F_R^{TAR})	24,5%	32,1%	33,7%	34,2%
E(F_R^{TOT}|(Drawdown & F_R^{TOT} < F_R^{TAR})	58,5	61,6	63,2	61,1
P(Switch between R e R+10)	17,2%	25,6%	28,1%	27,8%
Average year switch after R	5,3	4,8	4,6	5,1
P (0<F_{R+10}^{TOT}<F_{R+10}^{TAR} &F_t^{TOT}≥0; R<t<R+10)	3,9%	3,8%	2,9%	3,6%
E(F_{R+10}^{TOT}|F_{R+10}^{TOT} < F_{R+10}^{TAR})	4,3	4,3	5,2	5,1
P(Ruin)	3,4%	2,7%	2,7%	2,8%
Average year of ruin after R	9,1	9,1	9,1	9,0

Table 19: The Distribution Phase for Switch Strategy SC=27 with Bequest

We will limit ourselves to comment the results below the marked line. The probability **P(Drawdown & F_R^{TOT} < F_R^{TAR})** increases when we insert reserves in the switch model, however reaching lower values if compared to the equity strategy, as well as increasing is the average fund available to the individual at the beginning of the drawdown. The best results in terms of average fund compared to the equity strategy are obtained by the reserves calculation method introduced by us. The probability for the switch to take place between R e R+10 increases in the switch model with the introduction of reserves, while decreasing is the average number of years necessary in order for such switch to occur. The best results compared to the equity strategy are obtained by the reserves calculation method introduced by us. The probability of having a number of paths that reaches R +10 without switch and without ruin decreases with the introduction of reserves, while increasing is the average fund available at the occurrence of this hypothesis. The best results compared to the equity strategy are obtained by the reserves calculation method introduced by us. We finally analyze the probability of ruin. It decreases in the switch models with the introduction of reserves, reaching similar values for the two examined calculation methods and however achieving better results than the equity strategy. Regarding the number of years necessary in order for the ruin to occur, we can see that the switch strategy with and without reserve produces identical and however always better results than those of the equity strategy.

Also in this case we calculate the overall probability of failing the target and we report the results in the chart below:

	SC=27	**SC=27 with buffer**	**SC=27 with buffer (Method n.1)**	**100% equities**
Total probability of failing the Target	28,6%	13,8%	9,9%	6,4%

Table 20: Total Probability of not Reaching the Target for the Switch Strategy SC=27 with Bequest

The results clearly show that the introduction of reserves, as it was reasonable to expect, significantly lowers this probability; we obtain the best result by applying our calculation method, allowing to obtain an overall probability closer to that of the equity strategy, which however, during the distribution phase, has proved to be the least risky. The difference in terms of risk for the two strategies takes a value of about 0,35%. We must however underline that ,although the introduction of reserves, calculated by any method, allows to increase the average fund at maturity, however, in the switch strategy with SC =27 we are never able to match the one achieved with the "100% equities" strategy that, from this point of view, turns out to be the best.

Let’s see what happens when we consider the switch strategy with SC = 31 with reserves and bequest.

	SC=31	SC=31 with buffer	SC=31 with buffer (Method n.1)	100% equities
P (F_R^{TOT}<F_R^{TAR})	40,10%	37,40%	33,14%	34,20%
P (F_R^{TOT}<F_R^{TAR} |SF< 41)	18,94%	7,83%	5,05%	does not apply
P(No Drawdown & $F_R^{TOT} < F_R^{TAR}$)	14,00%	4,90%	3,30%	does not apply
E(F_R^{TOT}|(No Drawdown & $F_R^{TOT} < F_R^{TAR}$)	71,86	73,21	74,41	does not apply
P(Drawdown & $F_R^{TOT} < F_R^{TAR}$)	26,1%	32,5%	31,3%	34,2%
E(F_R^{TOT}|(Drawdown & $F_R^{TOT} < F_R^{TAR}$)	59,7	60,9	61,4	61,1
P(Switch between R e R+10)	20,9%	25,9%	24,9%	27,8%
Average year switch after R	5,0	5,1	5,0	5,1
P (0<F_{R+10}^{TOT}<F_{R+10}^{TAR} & F_t^{TOT}≥0; R<t<R+10)	3,1%	3,6%	3,6%	3,6%
E(F_{R+10}^{TOT}|$F_{R+10}^{TOT} < F_{R+10}^{TAR}$)	5,2	4,9	4,4	5,1
P(Ruin)	2,1%	3,0%	2,8%	2,8%
Average year of ruin after R	9,1	9,0	9,4	9,0

Table 21: The Distribution Phase for Switch Strategy SC=31 with Bequest

We always limit ourselves to comment on the results below the marked line. The probability **P(Drawdown & $F_R^{TOT} < F_R^{TAR}$)** increases when we insert reserves in the switch model, as well as increasing is the average fund available to the individual at the beginning of the drawdown. The best results compared to the equity strategy are obtained by using the reserves calculation method introduced by us. The probability for the switch to occur between R e R+10 increases as we introduce reserves, even if we never succeed in reaching the levels of the equity strategy. Instead, for all the compared strategies, rather similar are the results concerning the average number of years in order for that switch to occur. The probability of having a number of paths that reaches R +10 without switch and without ruin increases with the introduction of reserves, achieving results similar to those of the equity strategy, while decreasing turns out to be the average fund available at the occurrence of such an eventuality, as it never reaches the results of the equity strategy. Let's finally analyze the probability of ruin.
This appears to be increasing in switch models with the introduction of reserves, reaching values similar to those of the equity strategy. Regarding the number of years necessary in order for this ruin to occur, we see that our reserves calculation methodology is the one that produces the best result, removing in time the risk of bankruptcy.
Finally, also in this case, we calculate the overall probability of failing the target and we report the results in the table below:

	SC=31	SC=31 with buffer	SC=31 with buffer (Method N.1)	100% equities
Total probability of failing the Target	19,2%	11,5%	9,7%	6,4%

Table 22: Total Probability of Failing the Target for the Switch Strategy SC=31 with bequest

The results clearly show that the introduction of reserves, as it was reasonable to expect, significantly lowers this probability; the best result is obtained by applying our calculation method, allowing to get an overall probability, closer to the one of the equity strategy, which nevertheless, during the distribution phase, has proved to be the least risky. We must underline that, although the introduction of reserves, calculated by any method, allows to increase the average fund at maturity, however, in the switch strategy with SC = 31 we never succeed in matching the one achieved with the "100% equities" strategy, that, from this point ,turns out to be the best. In light of the obtained results we can say that, in realistic market hypothesis, the switch strategy for SC = 31 with the buffer calculated according to our new method allows to obtain, compared to the equity strategy, a safer strategy in the accumulation phase. However it cannot reach the levels of risk of the latter when the distribution phase is jointly considered, even if we note that the difference of risk between the two phases is highly reduced to a value of about 0, 34%.

7. Conclusions

In this work we have referred to the switch strategy suggested by the two Authors Arts and Vigna, whose main purpose was to devise a strategy allowing to take the higher risks in the younger age, lowering instead the risk as the R retirement approaches.

This strategy is based on the research of two excellent times, the time of first switch SC and the time of second switch SF. In the model proposed by the two Authors the identification of these two times is of extreme importance as they mark two key moments of change in the investment strategies of the Pension Fund. In fact, since the time 0, period in which the individual joins the pension scheme, up to SC, the contributions will be invested in equities and only later, from SC on, in bonds. At SC we therefore have two funds, the first obtained by investing the contributions from 0 to SC in equities and the other, that will be formed from SC on, with the contributions directly invested in bonds. Instead the second switch, that can take place from SC on, is exactly the moment when, under certain conditions, the equity fund formed earlier can be converted into bonds. The two Authors jointly considered the results of the proposed model both in the accumulation phase and in the distribution phase.

In particular and with reference to the distribution phase, the Authors established what follows:

- if the switch SF occurs before R, the individual, at the retirement time, must leave the scheme and, with the fund settled to it purchase an annuity;
- if this second switch does not take place between SC and R, then the Authors insert the possibility for the individuals to take the drawdown option;
- this means that funds keep remaining in the availability of the pension scheme, that will continue to invest them, though allowing the individual to annually withdraw a rate of pension P;
- yet this opportunity is not left forever since, at the reaching of 75 years of age, the individuals must compulsorily abandon the social security scheme by purchasing the annuity with the overall settled fund.

The peculiarity is that, by providing the possibility of the income drawdown option, it was also necessary a revision of the switch criterion for which now, once verified the suggested switch condition, one no longer provides for converting the equity fund into a bond fund, but will immediately provide for purchasing the annuity with the relative withdrawal from the fund. In fact, the strategy produced by the two Authors had strengths and weaknesses, also considering the basic assumptions taken into account[14] Our purpose was to modify the model in certain points and test its effectiveness when one changes some basic assumptions and introduces some new elements compared to the original work.We maintained the basic structure of jointly examining both the accumulation and distribution phase, although inserting the correlation between the financial assets. We also found new reserves calculation methods compared to the original model and finally we extended the analysis of the distribution phase, also including the hypothesis of income drawdown option with bequest payable to the member's relative. What we obtained is undoubtedly a model improved if compared to the previous one. When we talk about a better model than the previous one, we refer to the concepts of average fund at maturity and probability of failing the target at maturity. In particular, the best strategy that we obtained is the one providing SC = 31 with buffer with the new method of reserves calculation introduced by us, that allows us to obtain similar and in some cases better results than a "100 % equities "strategy.

In particular, on the basis of the achieved results, we reached the following conclusions.

a) With regard to the accumulation phase the strategy identified by us is less risky than the equity strategy, however, in terms of average fund at maturity, the latter is still the one with the best result even if the difference in the two cases considerably tails off, reaching an average value of about 0,10%.
b) With reference to the distribution phase this strategy, either providing or not the risk of mortality, can never reach the levels of the equity strategy which proves to be the less risky even if the difference between the two as in the accumulation phase is drastically reduced to an average level of approximately 0,35%.
c) We must also underline that all other measures of risk to our strategy (Standard Deviation, and Mean Shortfall) are still significantly lower than the equity strategy.
d) As noted in previous paragraphs a) b) c) we can therefore say that in more realistic market hypothesis the switch strategy proposed by the two Authors definitely improves the performance obtained by the originally formulated model.

[14] See Arts. Vigna, A Switch criterion for defined contribution pension schemes, Working paper 30/03, CeRP, pp.21 e 22.

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