Human Capital Valuation in Professional Sport

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Abstract

This paper is based on the premise that in professional sport, team earnings are driven by the performance of the individual athlete. It follows that the expected realizable value of each athlete must be accurately estimated in order to obtain a realistic estimation of team earnings. This paper undertakes such as assessment by measuring individual contribution to team revenue. For the superstar, revenue growth is modeled by a combination of an exponential growth model and a path based on a Brownian motion. The slow-growth performer's revenue growth is represented by an exponential growth model and two Brownian motions to account for increasing and declining revenue growth. The injured player's revenue growth is approximated by an early period of either superstar performance or slow-growth performance, followed by a Poisson distribution with discontinuities to account for periods of absence due to injury.

Keywords: optimization, sport, human capital, valuation, asset

1. Introduction

Neoclassical labor-market theory (Hicks, 1964) views the demand for labor as strictly utilitarian so that the value of labor is a function of its ability to produce output for the firm. It follows that as long as labor makes a positive contribution to providing a stream of future revenues for the firm, its continued employment is justified. In professional sport, team earnings, and indeed the team itself, would not exist without the individual athlete. It follows that the actual value of each athlete must be accurately estimated in order to obtain a realistic estimation of team earnings. Early studies have been based on current, rather than future contribution to team revenue. Scully (1974) identified playing time, on-field performance, star quality and experience as explanatory variables of current performance. Likewise, Quirk and Fort (1992) found that share of team at-bats, games started, and age had significant effects on current performance. We maintain that future contribution to revenue may not conform to current contribution in that absence due to injury and failure to meet expectations could result in an individual player's contribution to team revenue falling below the standard at which wage contracts have been negotiated. Accordingly, we propose a model to determine the true contribution of players to team revenue based on their contribution to TV contracts, sponsorships, ticket sales, concessions, apparel and stadium revenue.

2. Review of the Literature

McLaughlin (1994) set forth a generalized model of wage-setting in thin markets. Given scarcity rents, or large premiums above marginal revenue product paid to a dwindling number of professional athletes, the labor market for professional sports may be characterized as thin. If we assume that the player is desired by heterogeneous teams, the player's agent will engage in negotiation with the understanding that if the terms offered by this employer are unfavorable, he or she will accept a comparable offer from another team. Equilibrium will be achieved upon the acceptance of an offer that is equal to all other competing offers.

The final wage =

$$wi_j = w'i_j + \beta_j[M_{ij} - w'_{ij}] \quad (1)$$

wi_j = final wage,

$w'i_j$ = maximum opportunity wage or maximum wage offered by a rival team,

β_j = rent-sharing parameter

M_{ij} = marginal revenue of the player.

If we assume a rent-sharing parameter, $\beta > 1$, the two parties engaged in wage negotiation assume that for the duration of the contract, the marginal revenue will be higher than the opportunity wage. This may be true for a superstar such as Michael Jordan who achieved winning performances throughout his career and generated positive contributions to multiple revenue streams for the Chicago Bulls. This is certainly not true of the typical player who may age during the contract period or the player who has frequent absences due to injury. In those cases, a rent-sharing parameter that commences as being > 1, shifts to being < 0, or wages in excess of marginal revenue would ensue.

Aging results in deterioration of physical and mental capabilities. Schulz and Curnow (1998) observed that for a sample of baseball players engaged in the sport in 1965, performance peaked at age 27. Demiralp et al. (2012) extended these findings by observing that aging affects skills such as running speed and fielding which require less on-the-job learning and more physical ability with steady deterioration after the age of 28.

Conversely, hitting and pitching benefit from experience delaying aging's ill-effects by up to two years. We have concerns with the above stream of research from a methodological standpoint. The pooling of players of differing abilities and experience into a single sample may bias the peak age upwards for measures of the productivity paths of players over time. We endorse the grouping of players by ability and suggest categories of our own design as productivity paths and wages can only be rationally determined per player if we recognize the differences in ability that result in differences in contribution to the team's revenues.

The academic literature is sparse with regards to the impact of sports injuries on career length. Overfield (1989) observed that eight Major League Baseball players prematurely ended their careers after being injured by a pitched ball. Atkinson and Tschirhart (1986) observed that lower-body injuries per game were found to be highly significant in shortening career length with a 1% increase in such injuries predicting a 323-481% increase in failure time or decrease in career length in two of four regression models examined. In sum, withdrawal may be preceded by periods of absence due to injury. In our models, this is the assumption of injury effects that we employ.

3. An Economic Model for the Determination of Marginalrevenue Product Of Professional Athletes

The rent-sharing parameter defined in Section 2 (McLaughlin, 1994) serves to distinguish between three types of professional athletes. If $\beta > 1$, the scarcity rent paid to the athlete is justified in that the athlete earns a comparable wage offered by a rival along with a premium above the rival's wage to account for continuously increasing marginal revenue. This is the category of the all-time superstar. For athletes who are not all-time stars, but complete their contracts while gradually aging into retirement, a β that satisfies the condition $0 < \beta < 1$, may be envisioned. The scarcity rent to the athlete is partly justified with the comparable opportunity cost wage being supplemented with a premium to account for slowly growing marginal revenue. Slow growth continues until the athlete achieves peak performance. For athletes who are prone to injury that causes a succession of absences, resulting in eventual permanent withdrawal, a step function may be envisioned. Initially, $\beta > 1$, or $\beta < 0 < 1$. Subsequently, the player suffers his first injury necessitating an absence from the game. During the period of absence, if he continues to be paid, $\beta < 0$, wages will exceed his marginal revenue.

As wages are excessive, the team's focus may turn to finding a replacement. Following his absence, he returns to the field to play at a diminished level of ability, for we assume that he will never perform at his initial level of performance. A second injury further jeopardizes his future contribution to total team revenue. Once again, $\beta < 0$. Upon return, his skills are eroded even further. As such a sequence of events imposes an opportunity cost on the team which could have had continuous slow growth with the second category of athlete. The possibility of a trade becomes imminent, or the athlete is invalided out of the division (either to perform at a diminished level for a lesser-ranked team or retire from the game altogether).

3.1. The Superstar Athlete

1.Model Conceptualization: The marginal revenue product is the rate of change of total revenue with respect to the change in labor cost. Assuming that total revenue is represented by R and that the incremental labor cost of the superstar is L, the marginal revenue product for the ith athlete at time t is,

$$M_{it} = \partial R_t / \partial L_{it} \quad (2)$$

In order for the ith athlete to continue to be employed by the team, his marginal revenue product,

$\partial R_t / \partial L_{it} > 0$. Characteristics of this growth function include continuous upward movement of revenue with the assumption that the player does not peak throughout the period of the contract. We may conceive of this continuous growth function as an exponential growth function. Using the Malthusian population growth model with $R(t)$, Total Team Revenue, replacing Population Size, $P_i(t)$ and t representing time,

$$P_i(t) = P_0 e^{rit}$$

r_i, the growth rate of population, becomes g_1, the growth rate of team revenue in our sport model.

$$R(t) = R_0^{g1t}$$

Differentiating both sides with respect to t,

$$dR/dt = lnR_0 + g_1 t \quad (3)$$

Multiplying both sides by $P(i)$, the proportion of total revenue that represents athlete i's contribution,

$$P(i)(dR/dt) = P(i)lnR_0 + P(i)g_1 t$$

The analogous equation for labor is stated thus, with L, total labor cost for the team replacing total revenue in (2) and g_2, the growth rate of wages for athlete i replacing g_1,

$$P(i)(dL/dt) = P(i)lnL_0 + P(i)g_2 t \quad (4)$$

Both expressions (3) and (4) use an initial constant value for total revenue, R_0, and total labor cost, L_0 respectively.

Expression (3) represents the change in marginal revenue assuming a deterministic component of the change in R_t over time. The growth rate of this component, g_1, is predictable from historical values. For labor, similar relationships hold with expression (4) representing the change in the marginal wage, L_0, assuming a deterministic component of the movement of L_t along a sample path with a predictable growth rate of g_2. The stochastic component of marginal revenue or the marginal wage may be considered to have the characteristics of a Brownian motion including stationary, independent, normally distributed increments (Mikosch, 1999). Successive changes in revenue or wages over time are a sequence of random variables unrelated to past observations. We present total team revenue at time, t, as the following Langevin stochastic differential equation (see Mikosch, 1999, for a proof),

$$R_t = e^{ct}R_0 + \sigma e^{ct}\int_0^t e^{-cP(i)Rt} dB_{Rt}, \quad (5)$$

c = constant with values > 0,
σ = standard deviation of revenue changes over time,
B_R = Brownian sample path for the stochastic component of total revenue,
The corresponding expression for labor is

$$L_t = e^{ct}L_0 + \sigma e^{ct}\int_0^t e^{-cP(i)Lt} dB_{Lt}, \quad (6)$$

We may conceive of L_0 as the minimum market wage for athlete *i*. This wage is subject to a premium based on variance in total revenue and the Brownian sample path of total revenue. In other words, the final wage of the athlete is a vector based on the magnitude of $P(i)R_t$, the player's contribution to total revenue and direction defined by the Brownian sample path of team revenue, B_{Rt}.

At this stage, our challenge is to convert the variables defined in (1)-(6) to measurable quantities. We commence with g_1 and g_2, the growth rates of revenue and labor respectively. Revenue is expected to change by a larger amount than labor cost over the duration of athlete *i*'s contract. For an identical incremental unit of time, $g_1 > g_2$ for the team to justify continued employment of the athlete, as the incremental change in total revenue from all sources, i.e. broadcast revenue, sponsorship revenue, gate receipts, concessions and stadium fees, must increase more rapidly than the increase in wages over time. Do wages increase over time with contracts with fixed remuneration ? There is an implicit growth rate even in a contract of fixed duration with prespecified remuneration, assuming that there is expectation of renewal or migration to another team. The athlete hopes that his performance during the first contract period would be sufficiently successful to warrant renewal or offers from other teams at more favorable levels. In other words, a first five-year contract of $ 50 million should be followed by a second contract of $ 60 million, implying a growth rate of 5% per year.

The sources of total team revenue of broadcast revenue, sponsorship revenue, gate receipts, concessions and stadium fees are intercorrelated. A player who meets the criterion of $g_1 > g_2$ at *every* point in time must elevate performance to the level at which all of these sources of revenue continually increase as a direct consequence of his performance. It follows that only a one-of-a-kind superstar who has not reached his peak is capable of such an accomplishment. Therefore, a constrained optimization model for wage-setting for superstars must include the constraint that

$$g_{1n} > g_{1n-1} > g_{1n-2} > g_{1n-3} \ldots > g_{1n-t} > 0 \text{ where } t = \text{time } 0 \qquad (7)$$

If the formal model only considers the last two years of the player's contract, which are years *n* and *n-1*, expression (7) reduces to

$$g_{1n} > g_{1n-1} \qquad (8)$$

As $g_1 = \partial R/\partial t$ and $g_2 = \partial L/\partial t$, over time, $g_1 \rightarrow \infty$, $g_2 = \partial L/\partial t = 0$, or wages will achieve a maximum value. This maximum point could occur at a very high level, which may be beyond the budgeted capabilities of the team. Recognizing the potential of the athlete, the team may relax its budget constraint to increase the compensation of this superstar, with the expectation of increasing marginal revenue.

Relaxation of the budget constraint cannot continue indefinitely given the limit of a typical team's budget. Wages may increase as long as the change in final wage to the athlete is incrementally below the player's contribution to team revenue in the last year of the contract. Projections must be made of each type of revenue with or without the superstar by evaluating the contributions to revenues of contemporary superstars on other teams, the difference providing the future impact on revenue of the superstar. This difference (P_i) may be applied as a weight to each source of revenue listed in the objective function of the constrained optimization model below. Additional variables have been added as constraints to account for the player's contribution to competitive balance (Humphreys, 2002), teamwork, team quality, recruiting success and size of the market.

The abstract expressions (4) and (5) must be converted to measurable amounts of revenue, R_t,, and wages of athlete *i*, for any year, *t*. We reproduce (4) and (5) as (9) and (10) below, rewriting them with summations in lieu of the ordinary integral and a characteristic operator of a Brownian motion as ½Δ, where Δ denotes the Laplace operator.

$$R_t = e^{ct}R_0 + \sigma e^{ct}\sum_0^t e^{-cP(i)Rt}[i/2*1/[det(g)\sum_{I=1}^m \partial^2 x/\partial x_i(det(g)\sum_{j=1}^m g^{ij}\partial/\partial x_j] \qquad (9)$$

c = constant with values > 0,

σ = standard deviation of revenue changes over time,

$1/[det(g)\sum_{I=1}^m \partial^2 x/\partial x_i(det(g)\sum_{j=1}^m g^{ij}\partial/\partial x_j]$ = square of the Laplace-Beltrami operator in local coordinates, $[g^{ij}] = [g_{ij}]^{-1}$ in the sense of the inverse of a square matrix. The standard deviation of revenue changes over time and constant c may be measured implicitly (using a procedure similar to the computation of standard deviation of security returns and risk-free rate in the Black-Scholes model for the valuation of call options.

For example, revenue in (9) may be measured at two different points in time and the 2 unknown variables, σ and c solved through simultaneous equations. Expression (9) may be modified for each component of revenue, so that BR_t or broadcast revenue at time t will replace total revenue, R_t, etc. The corresponding expression for labor is,

$$L_t = e^{ct}L_0 + \sigma e^{ct}\sum_0{}^t{}_e{}^{-cP(i)}L_t[i/2*1/[det(g)\sum_{I=1}{}^m\partial^2x/\partial x_i(det(g)\sum_{j=1}{}^m g^{ij}\partial/\partial x_j] \quad (10)$$

c = constant with values > 0,
σ = standard deviation of labor changes over time,
$1/[det(g)\sum_{I=1}{}^m\partial^2x/\partial x_i(det(g)\sum_{j=1}{}^m g^{ij}\partial/\partial x_j]$ = square of the Laplace-Beltrami operator in local coordinates,
$[g^{ij}] = [g_{ij}]^{-1}$ in the sense of the inverse of a square matrix.

2. *Model Formulation*: The final constrained optimization model may be expressed as follows.

Maximize $P_i [(BR_n - BR_{n-1})/ (BR_{n-1})] x_1 + P_i [(SR_n - SR_{n-1})/(SR_{n-1})] x_1$
$+ P_i [(GR_n - GR_{n-1})/(GR_{n-1})] x_1 + P_i [CR_n - CR_{n-1}]/(CR_{n-1})] x_1 + P_i [(AR_n - AR_{n-1}]/(AR_{n-1})]$
$x_1 + P_i [(StR_n - StR_{n-1})]/(StR_{n-1})] x_1 + - \lambda [L_nx_1 + O_nx_2 - M_n]$
subject to
$g_{1n} > g_{1n-1}$ (11)
$g_{1n} > g_{2n}$
$g_{1\,n-1} > g_{\,2n-1}$
$P_iCB > 0$
$TD > 0$
$DQ > 0$
$P_iRS > 0$
$SZ > 0$
P_i = Weight of revenue component attributable to player i,
BR_n = Broadcast revenue at time n, the final year of athlete i's contract,
BR_{n-1} = Broadcast revenue at time $n-1$, the penultimate year of athlete i's contract,
x_1 = 1 for athlete i and 0 for all others,
x_2 = 1 for all other athletes and 0 for athlete i,
SR_n = Sponsorship revenue at time n, the final year of athlete i's contract,
SR_{n-1} = Sponsorship revenue at time $n-1$, the penultimate year of athlete i's contract,
GR_n = Gate receipts at time n, the final year of athlete i's contract,
GR_{n-1} = Gate receipts at time $n-1$, penultimate year of athlete i's contract,
CR_n = Concessions revenue at time n, the final year of athlete i's contract,
CR_{n-1} = Concessions revenue at time $n-1$, the penultimate year of athlete i's contract,
AR_n = Apparel revenue at time n, the final year of athlete i's contract,
AR_{n-1} = Apparel revenue at time $n-1$, the penultimate year of athlete i's contract,
StR_n = Stadium revenue at time n, the final year of athlete i's contract,
StR_{n-1} = Stadium revenue at time $n-1$, the penultimate year of athlete i's contract,
CB = Competitive balance is a measure of predictability of the outcome. Humphreys (2002) cautions that excessive predictability is detrimental to fan interest. It is measured as the ratio of the standard deviation of a team's win-loss percentages for all seasons to the standard deviation of a team's win-loss percentage per season,
TD = Teamwork dummy with a value of 1 if player i enhances the performance of the team, and 0 if he does not,
DQ = Quality of opposition dummy with a value of 1 if opposing teams are ranked within 2 rankings of player i's team and 0 if not,
RS = Recruiting Success is a measure of prior team success measured by the outcome of the past 3 years of recruitment,
SZ = Dummy variable to represent the size of the home market as teams serving large home markets may be able to replace star athletes more easily, with 0 = home market among the 3 highest ranked teams in player i's league (as a large home market increases the ease of replacement of star athletes, 1 if otherwise,
λ = Lagrange multiplier for budget constraint,
L_n = Wages for athlete i in year n, the final year of athlete i's contract,

O_n = Wages for athletes other than *i* in Year *n*, the final year of athlete *i*'s contract
M_n = Total labor budget for all athletes in year *n*.
$[g_{1n}, g_{1\ n-1}]$ = growth rate of the contribution to revenue from all sources in years *n* and *n-1* respectively, obtained by forecasting revenue per year and computing the annual growth rate from it,
g_{2n},
$g_{2\ n-1}$ = growth rate of wages in years *n* and *n-1* respectively,
This model yields values for x_1, the number of superstar athletes, and x_2, the number of non superstar athletes on the team.

3. *Solution Procedure:* We recommend the reverse of an exterior penalty function procedure (Avriel, 2003). We will add an incremental benefit to each additional item in the objective function in the progression to a local maximum. We assume that the decision-maker begins with the desire to maximize broadcast revenue, forecasting the player's contribution to broadcast revenue in the next to last year of the contract and the final year assuming a value for g_1 and g_2 and satisfaction of the condition that $g_1 > g_2$. Broadcast revenue may be presented as η so that the maximum revenue function at stage 1 is

$$\varphi(\eta) = |max(0,\eta)|^{\alpha} \qquad (13)$$

and broadcast revenue has the following function,

$$\zeta(\eta) = |\eta|^{\beta} \qquad (14)$$

where α and β are given constants, with values of 1 or 2. $\varphi(\eta)$ is maximized for broadcast revenue in Stage1. In Stage 2, a new total revenue function is created with γ for sponsorship revenue.
Sponsorship revenue has the function,

$$\zeta(\gamma) = |\gamma|^{\beta} \qquad (15)$$

and a new total revenue function $\varphi(\eta, \gamma)$ is maximized for both broadcast revenue and sponsorship revenue with respect to availability of wages. The process is repeated until gate receipts, concessions, apparel and stadium revenue denoted by σ, δ, ρ *and* μ respectively have been added to the objective function which in the final stage becomes $\varphi(\eta, \gamma, \sigma, \delta, \rho, \mu) = |max(0,\eta\ \gamma, \sigma, \delta, \rho, \mu)|^{\alpha}$ (16)
If x^{k*} is the optimal choice of athlete, there will be a sequence of points (x^k) which converges to the optimal choice.

3.2. The Slow-Growth Performer

1. **Model Conceptualization.** The second type of professional athlete is a person whoperforms consistently, who does not achieve the records of the superstar, yet is present regularly at games or is not prone to the spate of debilitating injuries that end careers prematurely. In terms of contribution to total team revenue, this individual does not arouse fan interest to the extent of being able to consistently increase team revenues over time. There will be years when a sufficient number of wins to the credit of this player will result in enhanced fan interest and in turn, increased broadcast revenue, gate receipts and concessions. In other years, fan interest may be diverted to other teams, so that on a year-to-year basis, each or all of the sources of revenue may be diminished. This athlete's contribution to team revenue has both a deterministic component and a stochastic component. The deterministic component is based upon predictable incremental contributions to revenue based upon past performance throughout the duration of the contract. Given its continuous upward trajectory, a modification of the Malthusian model will model its path.

$$R(t) = R_0^{g1t}$$

Differentiating both sides with respect to *t*,

$$dR/dt = lnR_0 + g_1t$$

Multiplying both sides by *P(i)*, the proportion of total revenue that represents athlete *i*'s contribution during the initial period of rapid growth,

$$P(i)(dR/dt) = P(i)lnR_0 + P(i)g_1t \qquad (17)$$

g_1 in this case may assume negative or 0 values. With 0 values, i.e. no growth in revenue, the revenue at time *t* equals the revenue at time *t-1*, with wages being equal to the minimum of competitive market wages.
The analogous equation for labor is stated thus, with *L*, total labor cost for the team replacing total revenue in (15) and g_2, the growth rate of wages for athlete *i* replacing g_1,

$$P(i)(dL/dt) = P(i)lnL_0 + P(i)g_2t \quad (18)$$

In equation (17), the early, rapid growth rate is modeled by the deterministic term, $P(i)(dR/dt)$. Later, two independent Brownian motions with paths. B_{at} and B_{bt} and standard deviations σ_i where i = 1 and 2 respectively, are introduced with one path approximating revenue increases and the second one approximating revenue decreases. The stochastic change in revenue may be stated thus (Mikosch, 1999),

$$R_t = R_0 + c \int^t R_t dt + \sigma_1 \int R_t dB_{at} + \sigma_2 \int^t R_t dB_{bt}, \quad (19)$$

with c and σ_i as constants.

The solution to (17) as presented by Mikosch (1999) for total revenue is as follows.

$$R_t = R_0 e^{[c - 0.5(\sigma 12 + \sigma 22)]t + (\sigma 1B_{at} + \sigma 2B_{bt}]} \quad (20)$$

Likewise, the corresponding solution to (17) for marginal labor costs is presented in (21).

$$R_t = R_0 e^{[c - 0.5(\sigma 12 + \sigma 22)]t + (\sigma 1B_{at} + \sigma 2B_{bt}]} \quad (21)$$

The change in revenue and change in marginal wages along 2 Brownian motion paths may be converted to measurable quantities using the procedure outlined for a single Brownian motion for superstar athletes in expressions (9) and (10). The growth rate of team revenue, g_1 will be positive, negative, or unchanged at different points in time. This suggests that the average value of g_1 must be positive over time, while there may be individual years with varying levels of g_1. Therefore, a constrained optimization model for wage-setting for slow growth performers must include the constraint that the average growth rate must be positive.

$$(g_{1n} + g_{1n-1} + g_{1n-2} + g_{1n-3} \ldots . > g_{1n-t})/n > 0 \quad (22)$$

If the formal model only considers the last two years of the player's contract, which are years n and $n-1$, expression (22) reduces to

$$[(g_{1n} + g_{1n-1})/2] > 0 \quad (23)$$

It follows that, at the limit, g_1 must assume a finite value as the player reaches his peak performance. The literature on aging has indicated that for skills that require agility such as running, stealing bases, and shooting hoops, performance peaks earlier than those for which experience dominates such as hitting or pitching. Therefore, depending upon the position played in baseball, the slow-growth player may peak towards the end of a six-seven year contract. The growth rate of the contribution of the individual player to team revenue will reach a finite limit, denoted by g_1 = Q, after which the player ceases to make a positive contribution to team revenue in any year. We assume that wages are concurrently growing at g_2. Ideally, in the period prior to the culmination of g_1 at Q (we will assume that it is n, the final year of the player's contract), wages should reach an upper limit as well or $g_{2n-1} = 0$. This would be the true final wage of the player. There is unlikely to be the need to relax the budget constraint to pay a larger salary to this player as the objective function of maximization of individual contribution to team revenue will not grow appreciably. The shadow price of labor will not be continuously positive, suggesting that paying exorbitant salaries to this category of athlete will never yield the increase in team revenue of the superstar. This conclusion may be reached by examining the rent-sharing parameter, β, of this athlete. Its values are $0 < \beta < 1$, while that of the superstar are always > 1 indicating a fractional increase marginal revenue over the increase in wages. Herein lies the cause of the misalignment of wages and marginal revenue to which we have alluded. When slow-growth players are paid at superstar rates, their increase in marginal revenue never justifies the increase in wages, or the change in marginal revenue < wages during the duration of the contract, causing substantial losses for the team.

In McLaughlin (1994) parlance, it is useful to track the shadow prices of labor throughout the player's contract. Given the expression, shadow price of labor = $P_j \, \partial(K_j,L_j)/\partial L_j$, demand elasticity for games and TV viewing increases as fans would seek alternative sources of entertainment if prices were raised.

Fan loyalty exists to a lesser extent with this category of athlete, so that fans will display eagerness to seek substitutes if prices rise. Prices may be assumed to be a constant. Further, as the change in output is unlikely to remain positive throughout time, instead of a sequence of positive shadow prices as with superstars, we may find a diverse combination of positive, negative, and unchanged shadow prices.

For example, in period 1, the shadow price of labor may be > 0 if the contribution to marginal revenue of this athlete increases total revenue, in period 2, the shadow price of labor may be < 0 if marginal revenue of this athlete decreases total revenue, while in period 3, the shadow price of labor may be = 0 if marginal revenue of this athlete leaves total revenue unchanged. An average shadow price over the three-year period would be more realistic to smooth out annual fluctuations. At the limit, such a three-year average may be used to cover periods n, $n-1$, and $n-2$ as the final period and $n-3$, $n-4$, and $n-5$ as the penultimate period. Such an average will be reflected in the formulation discussed next.

1.Model Formulation: We state a similar constraint optimization model to the superstar athlete which maximizes marginal revenue over time subject to a Lagrangian budget constraint, an average rate of growth of contribution to total team revenue, and an upper limit on wages,

Average forecasted revenue from broadcast revenue in the final period = $PriBR_{n, n-1,n-2}$ is the average of revenue from broadcast revenue earned in periods n, $n-1$ and $n-2$.
Average revenue from broadcast revenue in the penultimate period = $BR_{n-3,, n-4,n-5}$ is the average of revenue from broadcast revenue earned in periods $n-3$, $n-4$ and $n-5$.

Maximize $P_i\,[(BR_{n, n-1, n-2} - BR_{n-3,n-4,n-5})/(BR_{n-3,n-4,n-5})]\,x_1$
$+P_i\,[(SR_{n, n-1, n-2} - SR_{n-3, n-4, n-5})/(SR_{n-3, n-4, n-5})]x_1$
$+ P_i\,[(GR_{n,n-1,n-2} - GR_{n-3, n-4, n-5}]/(GR_{n-3,n-4,n-5})]\,x_1 + P_i\,[(CR_{n, n-1, n-2} - CR_{n-3, n-4, n-5})/(CR_{n-3, n-4, n-5})]\,x_1$
$+ P_i\,[(AR_{n, n-1, n-2} - AR_{n-3, n-4, n-5}]/(AR_{n-3, n-4, n-5})]\,x_1$
$+ P_i\,[(StR_{n,n-1.n-2} - StR_{n-3, n-4, n-5})/(StR_{n-3, n-4, n-5})]\,x_1 - \lambda[L_n x_1 + O_n x_2 - M_n]$
subject to
$(g_{1n, n-1, n-2} + g_{1n-3, n-4, n-5})/2 > 0$
$g_{1n, n-1, n-2} > g_{2n-3, n-4, n-5}$
$g_{1n, n-1, n-2} > g_{2n-3, n-4, n-5}$
$g_{1n-1, n-2, n-3} \leq Q$ (24)
$P_iCB > 0$
$TD > 0$
$DQ > 0$
$P_iRS > 0$
$SZ > 0$

P_i = Weight attributable to player i's contribution to team revenue,
$BR_{n, n-1, n-2}$ = Broadcast revenue during the final period of athlete i's contract,
$BR_{n-3, n-4, n-5}$ = Broadcast revenue during the penultimate period of athlete i's contract,
x_1 = 1 for athlete i and 0 for all others,
x_2 = 1 for all other athletes and 0 for athlete i,
$SR_{n, n-1, n-2}$ = Sponsorship revenue during the final period of athlete i's contract,
$SR_{n-3, n-4, n-5}$ = Sponsorship revenue during the penultimate period of athlete i's contract,
$GR_{n, n-1, n-2}$ = Gate receipts during the final period of athlete i's contract,
$GR_{n-3, n-4,,n-5}$ = Gate receipts during the penultimate period of athlete i's contract,
$CR_{n, n-1, n-2}$ = Concessions revenue during the final period of athlete i's contract,
$CR_{n-3, n-4, n-5}$ = Concessions revenue during the penultimate period of athlete i's contract,
$AR_{n, n-1, n-2}$ = Apparel revenue during the final period of athlete i's contract,
$AR_{n-3, n-4, n-5}$ = Apparel revenue during the penultimate period of athlete i's contract,
$StR_{n, n-1, n-2}$ = Stadium revenue during the final period of athlete i's contract,
$StR_{n-3, n-4, n-5}$ = Stadium revenue during the penultimate period of athlete i's contract,
CB = Competitive balance is a measure of predictability of the outcome. Humphreys (2002) cautions that excessive predictability is detrimental to fan interest. It is measured as the ratio of the standard deviation of a team's win-loss percentages for all seasons to the standard deviation of a team's win-loss percentage per season,

TD = Teamwork dummy with a value of 1 if player *i* enhances the performance of the team, and 0 if he does not,
DQ = Quality of opposition dummy with a value of 1 if opposing teams are ranked within 2 rankings of player i's team and 0 if not,
RS = Recruiting Success is a measure of prior team success measured by the outcome of the past 3 years of recruitment,
λ = Lagrange multiplier for budget constraint,
$L_{n, n-1, n-2}$ = Wages for athlete *i* in the final period of *i*'s contract,
$O_{n, n-1, n-2}$ = Wages for athletes other than *i* in the final year of *i*'s contract
$M_{n, n-1, n-2}$ = Total labor budget for all athletes during the final period of *i*'s contract.
$g_{1n, n-1, n-2}$,
$g_{1\ n-3, n-4, n-5}$ = growth rate of the contribution to revenue from all sources in the final period and the penultimate period of athlete *i*'s contract, obtained by forecasting revenue per year, averaging over two 3-year periods and computing the annual growth rate from it,
$[g_{2n, n-1, n-2}$
$g_{2\ n-3, n-4, n-5}]$ = growth rate of wages in years *n* and *n-1* respectively,
Q = growth rate of team revenue from player *i* in period n,

This model yields values for x_1, the number of slow growth athletes, and x_2, the number of non slow growth athletes on the team. The solution procedure is the maximization of marginal revenue product through the imposition of penalty functions and is identical to the solution of the superstar model.

3.3. The Injured Player

The injured player commences employment with either superstar or slow growth qualities, then, is absent due to injury the first time. His marginal revenue product = 0, if paid, wages > marginal revenue, initiating the desire to trade him. If unpaid, wages = marginal revenue, which does not benefit the team as marginal revenue must > wages for the player to contribute positively to team output. The desire to trade the athlete may emerge from this outcome as well, so that irrespective of the payment of compensation, the failure of the athlete to enhance marginal revenue during periods of absence may provide the impetus for the team to trade him. Upon return, the athlete returns to slow growth performance and then, if injured again, enters into a second period of absence when wages ≥ marginal revenue at which time he is traded. The periods of injury may be considered to approximate a Poisson distribution. Injuries are discrete events over a continuous interval of time, which by definition, is a Poisson distribution. The Poisson distribution is described by the arrival rate λ, of an injury and the service rate μ, which is the period of service (treatment) or absence from games.

Phase 1. The initial period may be conceived of as a period of superstar performance or slow growth as stated in expressions (1)-(19).
Phase 2. The first series of injuries (we prefer to consider them to be a series of injuries rather than a single injury) may be considered to occur at *n* subinintervals, $I_1 \ldots I_n$ of equal duration with a known mean of μ, and an occurrence of λ/n in each subininterval. The distribution is described by a Bernouilli process, which with a succession of injuries approximates to a Poisson distribution (Gullberg, 1997).

The probability of an athlete being injured with *k* injuries = $k = (\lambda^k e^{-k})/k!$ (25)

Upon injury, the athlete departs the game over a period of time termed the service rate, μ. For μ, the service rate, he receives treatment, and upon completion of the absence returns to the game.

No longer is he considered to be a superstar, but given the loss of his contribution to revenue, he has been relegated to slow growth status, if in slow growth mode in Phase 1, he may be facing removal. Using a Chernoff bound argument (Franceschetti et al., 2007), we may express the bounds of injury or the point at which the trade occurs. If the actual number of injuries (absences) *X* exceeds a threshold *x*, the probability of such a threshold being reached is given by (26). The team may decide that this probability is sufficiently high to trade the athlete to another team. $P(X \geq x) \leq e^{-\lambda}(e\lambda)^x/x^x$, *for* $x > \lambda$ (26)

1.Model Conceptualization: The superstar or slow growth periods use the model formulations developed in Sections and *III.A.* and *III.B.* respectively. As there is no revenue growth during the periods of absence, these models must be restated with finite period lengths to distinguish periods of performance from those of discontinuity. Assume that the first period is one of superstar growth, then there is an absence at time *n-4*, the athlete returns at *n-3*, leaves at *n-1* and returns at *n*. The final year of the first period is *n-4* and the penultimate period is *n-5.* For the second period of play, the final period is *n-2* and the penultimate period is *n-3*. For a slow-growth function, each period's revenue is usually a 3-year average. With discontinuity, a 3-year average cannot be computed, as the duration of each period may be only a single year. Therefore, we may relax the requirement of a 3-year average and employ a single-year average.

2.Model Formulation.

$$\begin{aligned}
\text{Maximize } & P_i\,[BR_{n-5} - BR_{n-4}/\,(BR_{n-5})]\,x_1 + P_i\,[SR_{n-5} - SR_{n-4})/(SR_{n-5})]\,x_1 \\
& + P_i\,[GR_{n-5} - GR_{n-4}]/(GR_{n-5})\,x_1 + P_i\,[CR_{n-5} - CR_{n-4}]/(\,CR_{n-5})\,x_1 + P_i\,[AR_{n-5} - AR_{n-4}]/(\,AR_{n-5})]\,x_1 \\
& + P_i\,[StR_{n-5} - StR_{n-4}]/(StR_{n-5})]\,x_1 - \lambda\,La_i\,[L_{n-5}x_1 + O_{n-5}x_2 - M_{n-5}] + P_i\,[BR_{n-2} - BR_{n-3}/\,(BR_{n-3})]x_1 \\
& + P_i\,[SR_{n-2} - SR_{n-3})/(SR_{n-3})]x_1 + P_i\,[GR_{n-2} - GR_{n-3})]/(GR_{n-3})\,x_1 + P_i\,[CR_{n-2} - CR_{n-3})]/(\,CR_{n-3})]\,x_1 \\
& + P_i\,[AR_{n-2} - AR_{n-3})/(\,AR_{n-3}]\,x_1 + P_i\,[StR_{n-2} - StR_{n-3}]/(StR_{n-3}]\,x_1 - \lambda[Lnx_1 + O_nx_2 - M_n]
\end{aligned}$$

subject to

$$\begin{aligned}
& g_{1n-5} > g_{1n-4} && (27)\\
& g_{1n-5} > g_{2n-4} \\
& g_{1\,n-4} > g_{\,2n-4} \\
& (g_{1\,n-2} + g_{1n-3})/2 > 0 \\
& g_{1n-2} > g_{2n-3} \\
& g_{n-2} > g_{2n-3} \\
& g_{1n-2} \leq Q
\end{aligned}$$

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