The Mathematical Limit of Longevity: A New Analysis

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Abstract

In this work we have taken our cue from the remarks made by the author B.M. Weon [18-22] in order to study the existence, or not, of a mathematical limit of longevity. Starting from the remarks made by the author, our purpose was to find a different survival function and therefore a new model (Pasqualitto Model) to replicate the analysis he conducted with reference to the mortality tables of the Italian population for the period 1950-2006. The choice of a different function was carried out in order to identify a model which, with reference to the analysis conducted by us, would better fit to the actual data, especially in relation to older ages, ages that, from a demographic point of view, have always been critical to be represented and synthesized by a suitable model.

Key Words: mathematical limit, longevity trend, Pasqualitto Model, Italian population.

Introduction

The variation of longevity over time has always been a big brain teaser for scientists and academics who deal with this topic, being a theme that encompasses and affects many areas, from the purely medical to that of applied mathematics. Indeed, the evolution of longevity is the basis of many actuarial calculations, and cannot be underestimated for the economic consequences that may arise at the expense of several insurance companies, as a result of an incorrect and not adequate estimate of the phenomenon itself. At present we can say that many of the traditional methods used for the study of the mortality evolution over time, such as for example the Gompertz model [4], the Weibull model [17], the Heligman & Pollard model, namely Quadratic and Logistic-type models, have demonstrated a low capacity to adapt to the real data of the population, especially with regard to older ages as amply demonstrated by some authors [13], [25]. Among the models listed above we can say that those who currently find greater application [3], are the Gompertz model (Gompertz, 1825) and the Weibull model (Weibull, 1951). In particular, from recent studies [2], [3] emerges that the Gompertz model is most commonly used to describe biological systems, while the Weibull model is more commonly used for technical devices.

Traditionally, the law of survival described by the Gompertz model is the following [23]:

$$S(x) = e^{\left[\frac{a}{b}\left(1-e^{(bx)}\right)\right]} \qquad (1)$$

where a and b are constant, while for the Weibull model we have [10]:

$$S(x) = e^{\left[-\left(\frac{x}{\alpha}\right)^{\beta}\right]} \qquad (2)$$

with α and β constant and β also called shape parameter. From the models described above clearly emerges that the mortality rate increases exponentially in time with increasing age (indicated with x), with relation to older ages. However, empirical analysis has shown that, just with reference to older ages, these models show the most obvious gaps. This discrepancy between theoretical models and real data is precisely the crucial point that gave rise to numerous debates between scientists and academics in order to identify the best models that could more correctly describe the entire survival curve. In fact, what seems to be more difficult to interpret, is just the phenomenon of rectangularization, that still seems to have been only partially satisfactorily explained as based on interpretations and results deriving from inappropriate models [1]. In order to overcome this difficulty Weon, in his various works [18-22], developed a model with a β(x) shape parameter, x age-dependent, able to describe the survival function, the curve of mortality and, more in detail, the decrease of the same in relation to older ages.

The Weon Model

The Weon model directly derives from the Weibull model and, for the survival function, assumes the following expression:

$$S(x) = e^{\left[-\left(\frac{x}{\alpha}\right)^{\beta(x)}\right]} \qquad (3)$$

From which emerges that it is fully described by only two parameters: α (also called life characteristic) and β(x), the shape parameter that, unlike the Weibull model, is no longer constant but variable, depending on age. Let's now more fully describe these two parameters. α was defined as the characteristic life and is usually interpreted as the age for which is:

$$S(\alpha) = e^{-1} \qquad (4)$$

In the Weon model we have that $S(\alpha) = e^{-1}$= 36, 79%, which means that at the age about the 37% of the population under consideration is still alive.

β(x) instead, shows the shape parameter of the survival function and is known to be a function of age. This assumption derives from Weons' direct observation of the survival function which, with increasing age, shows the following trends:

1) A rapid decline in survival during the first 5 years;
2) A relatively constant decline with increasing age;
3) A sharp decrease in the years near death.

As noted by Weon, the trend described in point 1) for the survival function is similar to that described by the Weibull function when $\beta < 1$. The trends described in paragraphs 2) and 3) seem instead to describe the survival function according to the Weibull model when $\beta \gg 1$, that's why this suggested to the author the hypothesis of the presence of a shape parameter that was not constant, but age-dependent. More in detail the $\beta(x)$ shape parameter whose expression can be directly derived from the survival function through the relation:

$$\beta(x) = \ln\,(-\ln S))/\ln\,(x/\alpha) \qquad (5)$$

Actually indicates the rectangularization of the survival function because the higher is the value of this parameter, the more the survival function assumes a rectangular shape. It should also be noted that the value of $\beta(x)$ tends to be infinite when the x age tends to α or, in other words, when the $\ln\,(x/\alpha)$ denominator tends to 0, this mathematically means that α represents a singular point for the $\beta(x)$ function. Weon has shown in his works that if you can find a suitable function able to describe the behaviour of the $\beta(x)$ parameter (except at the point of singularity α), then you can mathematically calculate both the survival function and the mortality function. Dwelling on a purely practical aspect, the description of the survival function as a whole turns out to be quite complex, that's why Weon, in order to consider and study the rectangularization of the same, divided it into three distinct phases. The first phase is called *developmental phase* (below 30 years), where one normally observes a significant increase of the $\beta(x)$ shape parameter. The second phase, called *mature or middle age phase*, between 30-70, instead shows how the expression of $\beta(x)$ can be adequately described by a linear- type expression. The third phase called *senescence phase*, that goes from the characteristic α age to the extreme age, can instead be well represented by a quadratic function:

$$\beta(x) = \beta_2 x^2 + \beta_1 x + \beta_0$$

whose coefficients were determined by a regression analysis in the plot of the shape parameter curve.
By virtue of what pointed out above, the first derivative of $\beta(x)$ is obtained as follows:

$$\beta'(x) = 2\beta_2 x + \beta_1$$

And represents a result that will be useful in the calculation of the mortality function. Let's consider what are the main differences about the wording of the mortality law, according to the Gompertz model and the Weon model.

In the Gompertz model you assume that the mortality rate increases exponentially in time when age increases in the senescence phase, however, some authors have shown that is not demonstrable with certainty whether the mortality level increases or decreases in the older ages [14], [16], [12] and [7].

With the suggested model, Weon, by means of the quadratic expression given to the shape parameter, is instead able to demonstrate that the mortality rate necessarily decreases after a ρ plateau, approximately reaching 0. More in detail this ρ plateau represents the maximum value of the mortality function after the characteristic life α and can be thus identified:

$$\frac{d\mu(x)}{x} = 0 \text{ at } x = \rho \tag{6}$$

In general Weon has shown that his model approximates that of Gormpertz when $\beta(x) \propto x$, on the contrary, the mortality rate differs from that provided for by the Gompertz model when $\beta(x)$ hasn't got a linear-type behaviour. Therefore Weon suggests that the $\beta(x)$ parameter can actually be a measure of the deviation from the Gompertz model at older ages. In particular, as it was described, the Weon model is nothing more than a generalization of both the Gompertz model and the Weibull model. More in detail, the Gompertz model is a special case of the Weon model, in the hypothesis that $\beta(x)$ is linear and the Weibull model is also a special case of the Weon model when the shape parameter is constant. Once determined the mathematical function able to approximate the shape parameter, we have seen that, through the Weon model, one can mathematically calculate both the survival function and the $\mu(x)$ mortality function.

Remembering indeed that $\mu(x) = -\frac{d}{dx}\ln S$, for the Weon model we will have that:

$$\mu(x) = -\frac{d}{dx}\ln S = \frac{d}{dx}\left[\left(\frac{x}{\alpha}\right)^{\beta(x)}\right] \tag{7}$$

or, in perfectly equivalent terms:

$$\mu(x) = \left(\frac{x}{\alpha}\right)^{\beta(x)} \cdot \left[\frac{\beta(x)}{x} + \ln\left(\frac{x}{\alpha}\right) \cdot \frac{d}{dx}\beta(x)\right] \tag{7.1}$$

Once known the survival function and the mortality function you can calculate the density function as follows:

$$f(x) = S(x) \cdot \mu(x) \tag{8}$$

That describes the distribution of life probability for a given population. Now let's see how Weon proceeds to calculate the limit of longevity. The most common assumption underlying this idea is that there should be some w age limit beyond which there are no more survivors; this can be represented by one of the following three expressions $f(x) = 0$, $S(x) = 0$, $\lim_{x\to w}\mu(x) = \infty \quad (x \geq w)$

However, what Weon demonstrated with his model, suggests that the survival function will never be equal to 0, even if it takes extremely low values at older ages, whereas the mortality function may be 0 in correspondence to the maximum longevity, therefore the three conditions seen above may simply be reduced to two, namely to $x = w$, $f(x) = 0$ or $\mu(x) = 0$. On the other hand what the author states is that the rate of growth of the survival function defined as:

$$-\frac{d}{dx}S(x)$$

Should be 0 at the maximum longevity, so ultimately the author assumes that the maximum longevity besides being expressed by the two conditions stated above, can also be expressed by this third condition, namely:

$$-\frac{d}{dx}S(x) = 0$$

According to these remarks Weon identifies the maximum longevity “as the moment in which the trajectories of the survival function levels off or, in other words, the trajectories of mortality become null”. At this point let’s illustrate, the procedure employed by Weon in order to define the mathematical limit of longevity. Let’s return to the expression for $\mu(x)$ indicated by (7.1). In order for his analysis, Weon separately considers the effects of the two mathematical terms on mortality:

$$A = \left(\frac{x}{\alpha}\right)^{\beta(x)} \qquad B = \left[\frac{\beta(x)}{x} + \ln\left(\frac{x}{\alpha}\right) \cdot \frac{d}{dx}\beta(x)\right]$$

More in detail, the result must be that $\mu(x) > 0$.

Being the term:

$$A = \left(\frac{x}{\alpha}\right)^{\beta(x)} > 0$$

Then the result will have to be:

$$B = \left[\frac{\beta(x)}{x} + \ln\left(\frac{x}{\alpha}\right) \cdot \frac{d}{dx}\beta(x)\right] > 0$$

namely:

$$C = \frac{d}{dx}\beta(x) > D = -\frac{\beta(x)}{x\ln\left(\frac{x}{\alpha}\right)} \qquad (9)$$

Having approximated the value of $\beta(x)$ with a quadratic function makes it easy to solve the equation stated above. Through this approximation Weon shows that:

- The coefficient of the β_2 quadratic term "supports the bending of the shape parameter".
- β_2 is directly associated with the deceleration of mortality.
- β_2 Allows to evaluate the longevity limit because it is the bending with respect to the age of the first derivative of $\beta(x)$.

Weon’s results show that the C term is decreasing with the x age and its bending becomes steeper over the years. On the contrary the D term increases with age and thus the steeper becomes the C term, the more reduced is the longevity limit over the years. The use of this approach, which arises from a comparison between the C and D terms, suggested to Weon the idea that there is a w longevity limit, linked to β_2 by an exponential-type relationship.

Our Model: The Pasqualitto Model

Starting from the analysis and conclusions reached by Weon, we carried out a more detailed analysis of the Italian situation. In particular, we made reference to the data provided by the Human Mortality Database website (www.mortality.org) for the Italian population (both sexes) from 1950 to 2006. The length of the time series under consideration is the result of the decision to avoid that our analysis could be affected by anomalous values such as those produced by the world wars on mortality. The S survival probability is defined as a $\frac{l_x}{l_0}$ fraction of the l_x number of survivors out of 100000 l_0 persons in the original life tables. While maintaining the quest for the longevity limit according to the procedure described by Weon, our goal was to find a suitable model that would fit better to real data, especially in relation to older ages. Let’s then proceed to analyze what suggested by us. With reference to the time series examined by us, we have found that the S(x) function suggested by Weon, while adapting well to the characteristics of our population, could still be susceptible to further improvements, especially with reference to older ages. Our first purpose was therefore to find a different survival function that would allow a quadratic approximation for older ages, better than the one suggested in the Weon model.

After several attempts at modelling and subsequent testing, we identified the following representation for the survival function:

$$S(x) = \frac{ke^{\left[-\left(\frac{x}{\alpha}\right)\right]^{\beta(x)}}}{1 + ce^{\left[-\left(\frac{x}{\alpha}\right)\right]^{\beta(x)}}} \qquad (10)$$

with:

k and c constant, determined by appropriate boundary conditions;
$\beta(x)$ age-dependent shape parameter;
α characteristic life;

You should note that for c =0 and k =1 our survival function coincides with that of Weon. The first step we made was to obtain $\beta(x)$ starting from the hypothesized survival function. Through a series of simple mathematical passages we have:

$$\beta(x) = \frac{\ln\left[-\ln\frac{S(x)}{(k - S(x)c)}\right]}{\ln\left[\left(\frac{x}{\alpha}\right)\right]} \qquad (11)$$

As already pointed out by Weon in his works, also in the Pasqualitto Model we see that the $\beta(x)$ function tends to infinity when x age tends to α or, in other words, when the $\ln(x/\alpha)$ denominator tends to 0, therefore also in our model α, the characteristic life, represents a singular point for the $\beta(x)$ function. Having established this, let's now proceed to calculate α. We define the boundary conditions on $S(x)$ and $\beta(x)$ in order to identify the characteristic life. For S(x) we have that:

if x = 0 then S (0)=1 and therefore k = (1+c);

For $\beta(x)$ we have to verify the following conditions:

$$-\ln\frac{S(x)}{(k - S(x)c)} > 0$$

and

$$\frac{S(x)}{(k - S(x)c)} > 0$$

Therefore:

$$0 < \frac{S(x)}{(k - S(x)c)} < 1$$

That is translated into:

$$\begin{cases} k - Sc \neq 0 \\ k - Sc > 0 \\ S < k - Sc \end{cases} \rightarrow \begin{cases} k - Sc > 0 \\ \\ S < k - Sc \end{cases} \rightarrow \begin{cases} k > c \\ \\ S(1 + c) < k \end{cases}$$

Using the condition on S and the two previous conditions, we have that:

$$\begin{cases} k = 1 + c \\ k > c \\ S(1 + c) < k \end{cases} \rightarrow \begin{cases} k = 1 + c \\ \\ S(1 + c) < k \end{cases} \rightarrow \begin{cases} k = 1 + c \\ \\ S < 1 \end{cases}$$

So for x different from 0, it follows that k = 1 + c.
For x = α the (10) is equal to:

$$S(\alpha) = \frac{ke^{-1}}{1 + ce^{-1}} \qquad (11.1)$$

This value is equal to 0,368 (36.8%) in the Weon model, to which corresponds k =1 and c = 0. We have already said that the value of the survival function calculated for x = α represents the percentage of survivors existing exactly at that age. By suitably choosing the value of c (and therefore k) in our survival function is possible to determine the value of the characteristic life. We decided to set c = 1, and then k = 2, therefore the survival function becomes:

$$S(x) = \frac{2e^{\left[-\left(\frac{x}{\alpha}\right)\right]^{\beta(x)}}}{1 + e^{\left[-\left(\frac{x}{\alpha}\right)\right]^{\beta(x)}}} \qquad (12)$$

In our model the survival function in correspondence to the α characteristic life is:

$$S(\alpha) = \frac{2e^{-1}}{1 + e^{-1}} \qquad (13)$$

against a value of 36.8% identified by the Weon function. We have already said that the value of the survival

$$S(\alpha) = 53{,}78\% \qquad (13.1)$$

function calculated for x = α represents the percentage of survivors exactly existing at that time.

We should also remark how the value of the survival function assumes for our model values closer to another indicator often used in literature for demographic analysis and purposes, such as that of the average life which we have for a value of x such that S(x) = 50% . This has been another element that helped us in the choice of the constants c and k.

As already mentioned by Weon, in order to identify the value of the characteristic life for the time series studied by us, you just need a careful observation of both the graph of the survival function and the graph of shape parameter, for which is possible to immediately identify the behavior that the characteristic life has had over the years, representing for this a singular point, (see Figure 1).

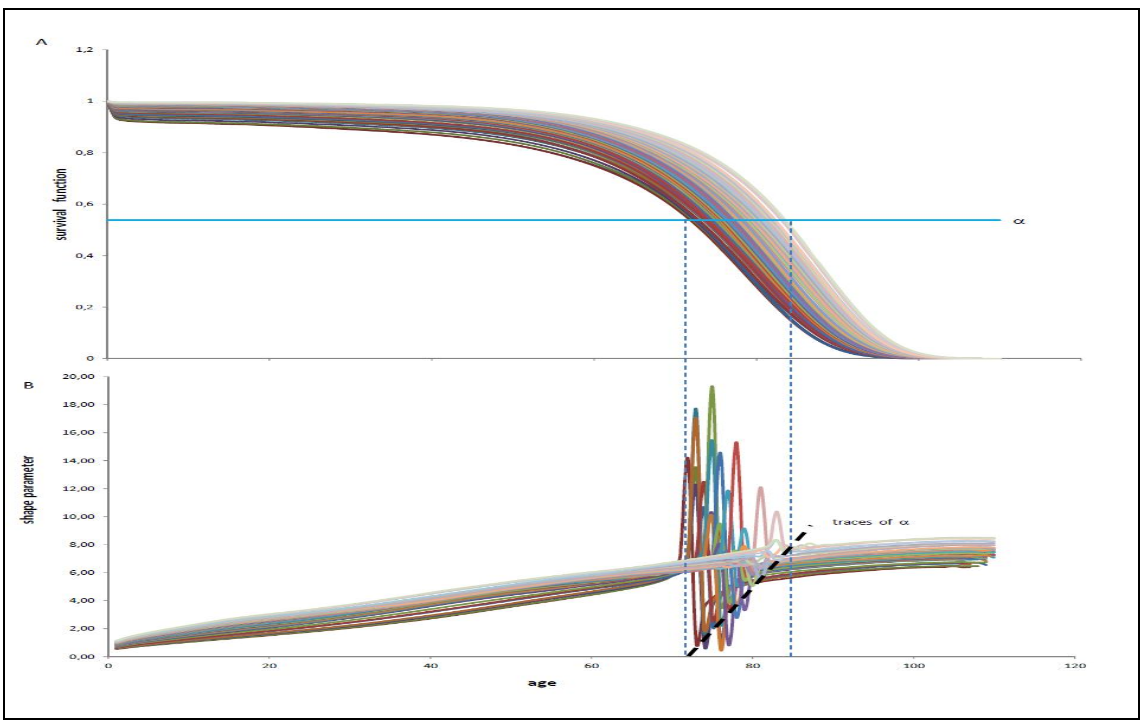

Figure 1: Trend of characteristic life 1950-2006

What can be seen from Figure 1 is how characteristic life α has gradually increased over time and this is undoubtedly attributable to the improvement occurred in the economic and social conditions and to medical progress in general, which have led to an increase in life expectancy over the years.

To determine the numerical value of α, for the different years, we have followed the following procedure. From the graphical analysis of the survival functions we have found that the value of the characteristic life is between 65 and 90 years. In this age range the survival function can be approximated by a function of second degree:

$$S(x) = ax^2 + bx + c \tag{14}$$

The a, b and c terms were obtained through a polynomial regression of each survival curve in the age 65-90. Such approximation involves an extremely low error for all the examined years. By way of example, we have reported in Figure 2 the index R^2 for some years of the time series under consideration.

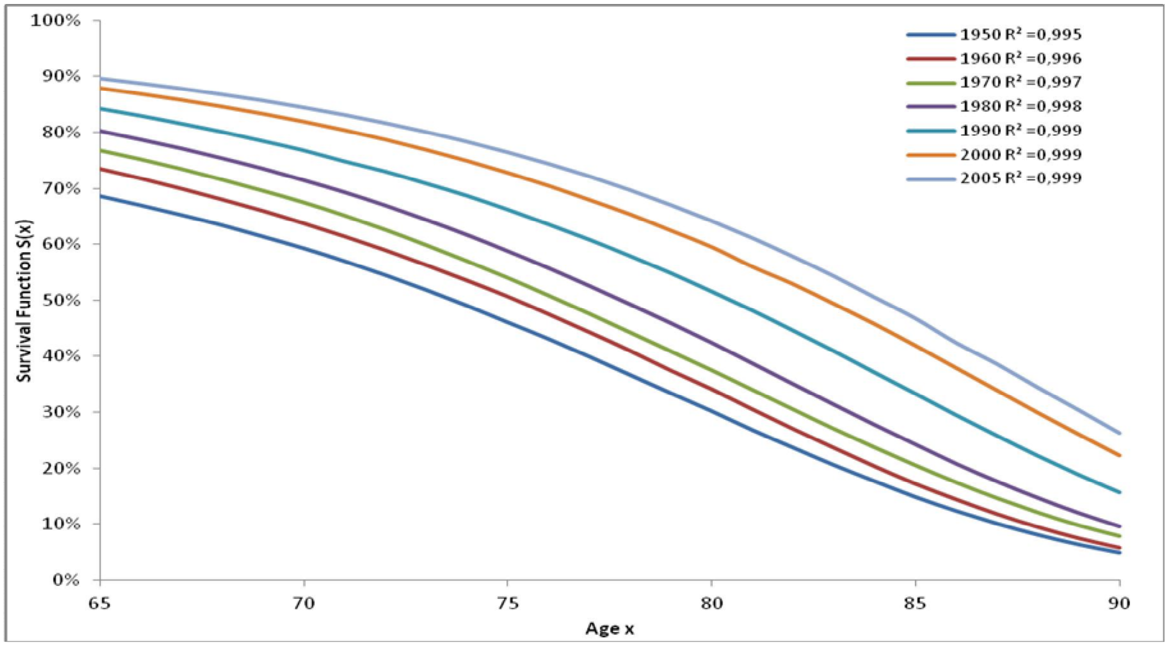

Figure 2: Fitting of the second degree functions compared to the survival functions

According to what defined above, once known the value of the survival function in correspondence of α, we were able to calculate α, thanks to equality:

$$a\alpha^2 + b\alpha + c = \frac{2e^{-1}}{1 + e^{-1}}$$

Of the two solutions obtained, we obviously discarded the one outside the interval 65-90, see Figure 3.

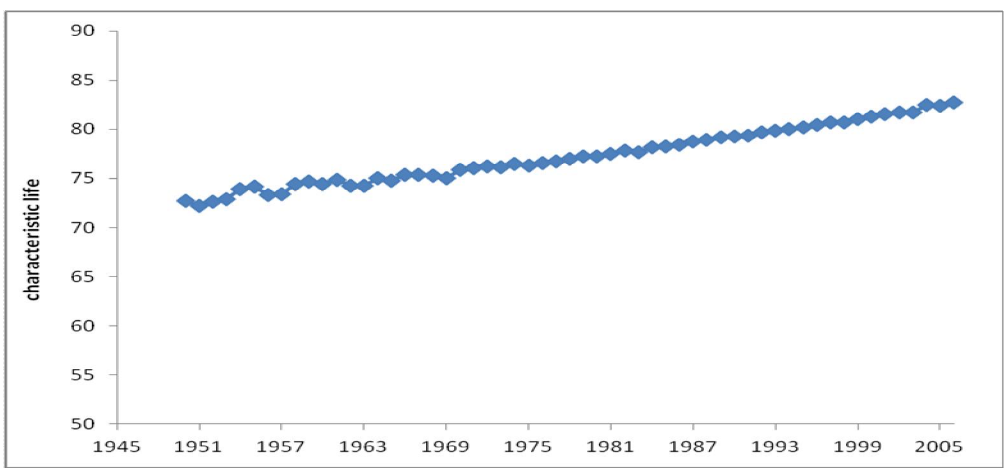

Figure 3: Characteristic life 1950-2006

Once α is found is therefore possible to proceed with the calculation of the β(x) shape parameter in correspondence to the different age groups, as identified by the formula (11). Let's now examine, also for our model, the calculation of the limit of longevity:

Once known the survival functions, we provided for calculating the $\mu(x)$ intensity of mortality through the following relation:

$$\mu(x) = -\frac{d}{dx}\ln\ (S)$$

For our survival function, after appropriate calculations we will have that:

$$\mu(x) = \frac{1}{\alpha}\cdot\frac{\beta'(x)\cdot x + \beta(x)}{1 + e^{\left[-\left(\frac{x}{\alpha}\right)\right]^{\beta(x)}}} \qquad (16)$$

From the formula (16) it is plain that, in order to calculate the intensity of mortality, we need the first derivative of the shape parameter. The problem that arises now is to find a function able to approximate the trend of $\beta(x)$. Let's go back to what Weon said about it in his works. We observed the behaviour of the shape parameter immediately after the characteristic life α and up to the extreme age, being α a point of singularity for $\beta(x)$. The result of the graphic analysis is that the trend of the $\beta(x)$ shape parameter, as for the Weon model, can be described by a quadratic relationship:

$$\beta(x) = \beta_2 x^2 + \beta_1 x + \beta_0$$

in which the values of β_0, β_1, β_2 were obtained by making a quadratic regression on the values of β from age x > α.

It must be highlighted that the quadratic approximation for the $\beta(x)$ is valid for x > α. The α of our model assume lower values than those calculated by Weon because:

$$S(\alpha)(WEON) < S(\alpha)(PASQUALITTO).$$

Nevertheless the fitting of the function $\beta(x)$ and S(x) led to a better approximation and thus to very precise calculations. By way of example we reported in Figure 4 the index R^2 for some years of the time series considered by us for the function S (x) for x> α.

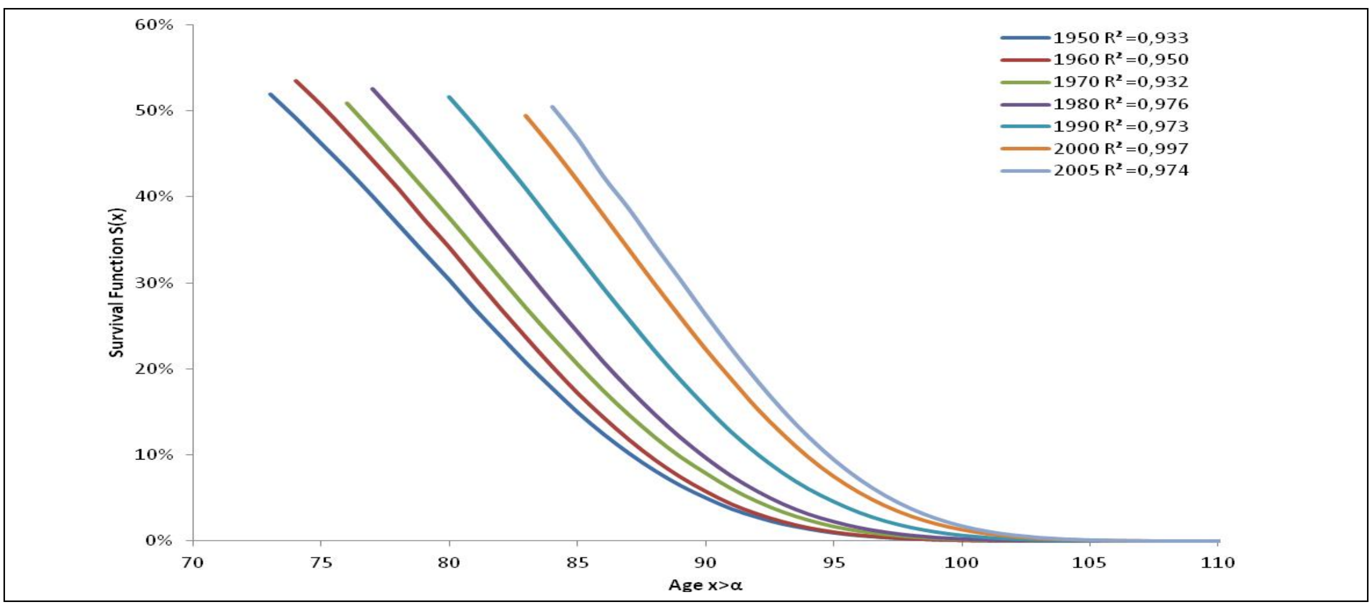

Figure 4 : Fitting of the survival functions for x>α

By virtue of the expression of$\beta(x)$, it follows that:

$$\beta'(x) = 2\beta_2 x \quad + \beta_1$$

Therefore, once we know $\mu(x)$ and $\beta'(x)$, we have all the elements necessary to calculate the value of the w extreme age. First it will always have to be: $\mu(x) > 0$, so let's see what are the necessary conditions for this to occur. With reference to the formula (16) is always verified that:

$$\frac{1}{\alpha} > 0$$

$$1 + e^{\left[-\left(\frac{x}{\alpha}\right)\right]^{\beta(x)}} > 0$$

Therefore $\mu(x) > 0$ if $\beta'(x) \cdot x + \beta(x) > 0$ that is if:

$$\beta'(x) > -\frac{\beta(x)}{x} \qquad (17)$$

Let's replace $\beta'(x)$ and $\beta(x)$ with their expressions and we will have:

$$3\beta_2 x^2 + 2\beta_1 x + \beta_0$$

from which:

$$w_{1,2} = \frac{-2\beta_1 \pm \sqrt{4\beta_1{}^2 - 12\beta_2\beta_0}}{6\beta_2} \qquad (18)$$

To this purpose we report some graphs illustrating the obtained results:

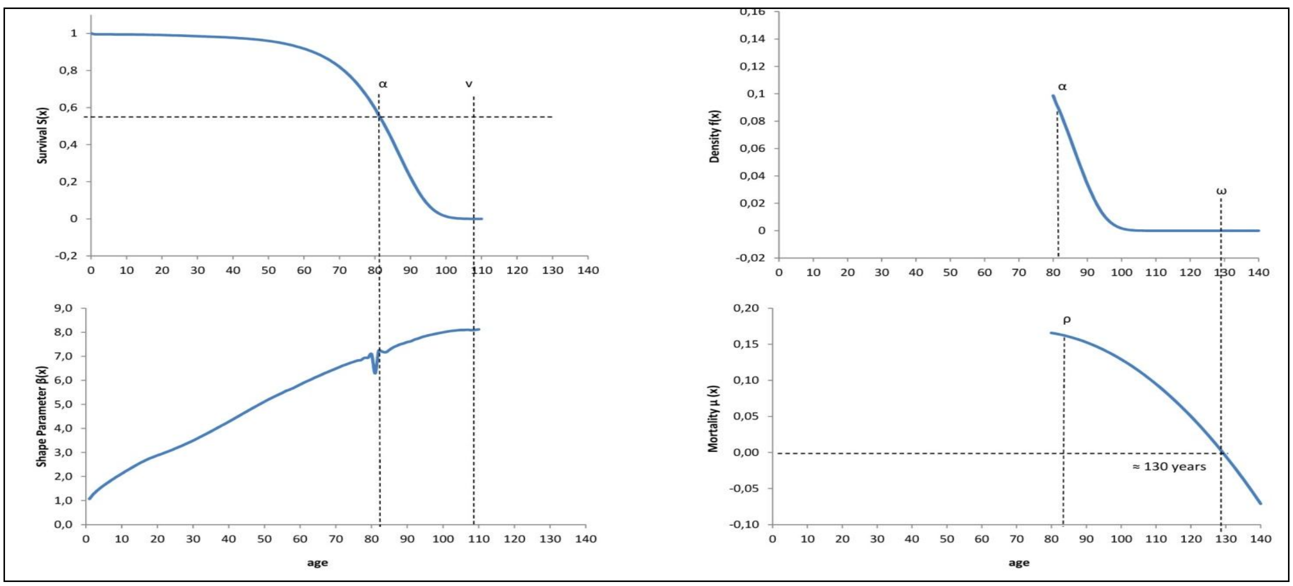

Figure 5: Demographic trajectories and singular points for italian population, 2000

In Figure 5 we reported some specific points that are found in our analysis. We already defined the characteristic life and how this can be highlighted by a graphical analysis of the survival function and by the corresponding graph of the shape parameter, representing for this a point of singularity. We also note the presence of two particular points:

- a point v that represents the maximum value assumed by the shape parameter after the characteristic life;
- a point ρ that represents instead the maximum value of the mortality function after the characteristic life.

In particular, in the two graphs on the right, we reported the density function and the mortality function only after the α characteristic life, age from which we provided for calculating the longevity limit. By the study of these two trajectories is possible to define the ultimate limit of longevity, being this the point where $f(x) = 0$ or $\mu(x) = 0$. For the year 2000 we have that w ≈ 130 years. Taking now into consideration the entire time series examined by us, we said that the longevity limit is recognizable when the following condition occurs:

$$C = \beta'(x) > D = -\frac{\beta(x)}{x}$$

We provided for a graphical representation of the above mentioned points in order to identify how this limit has changed over the years. Note the following figure:

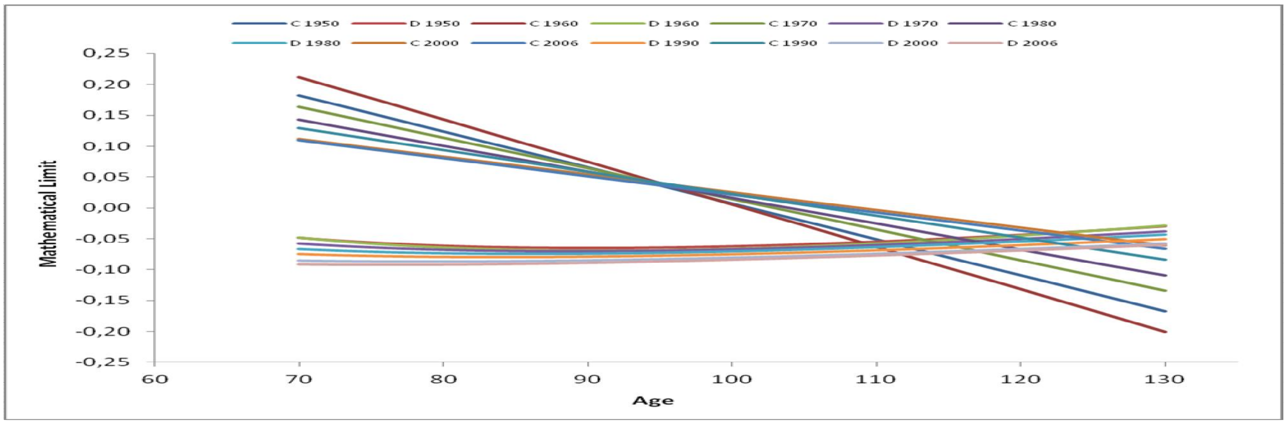

Figure 6: Mathematical Limit, 1950-2006

What is evident is that the C term tends to decrease with increasing age, on the contrary the D term increases (in absolute value) with increasing age. The specific trends of these two points over the years make us therefore say that the limit of longevity has decreased over time. The useful information for the calculation of the limit to longevity is provided by the quadratic coefficient of the β_2 shape parameter, which embodies a lot of information. It is known that the quadratic coefficient describes the curvature of the shape parameter and this curvature represents the deceleration of mortality over the years. It also represents the bending with respect to the age of the first derivative of $\beta(x)$, for which it also provides a measure of the velocity with which the $\beta(x)$ parameter varies according to age in a given reference year and therefore represents the velocity with which the phenomenon of the rectangularization of the survival function has changed over time.

In confirmation of what has been said we have represented the slope of the shape parameter in the middle age, see Figure 7. The figure shows that over time there has been a sort of saturation of this slope, and this may suggest the presence of a possible limit to the phenomenon of longevity. Namely, seems to exist a limit distribution that over time the curve of mortality can achieve but not exceed.

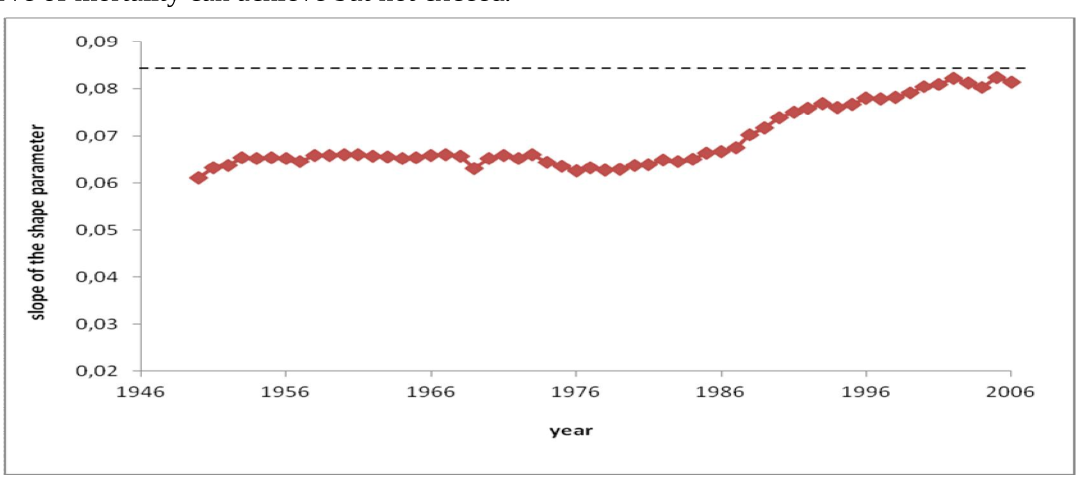

Figure 7: Slope of the shape parameter

In order to proceed then to the numerical calculation of the limit of longevity (ultimate limit of longevity), we calculated through (18) the value of w for each year under consideration and we represented the trend as a function of β_2, (see Figure8).

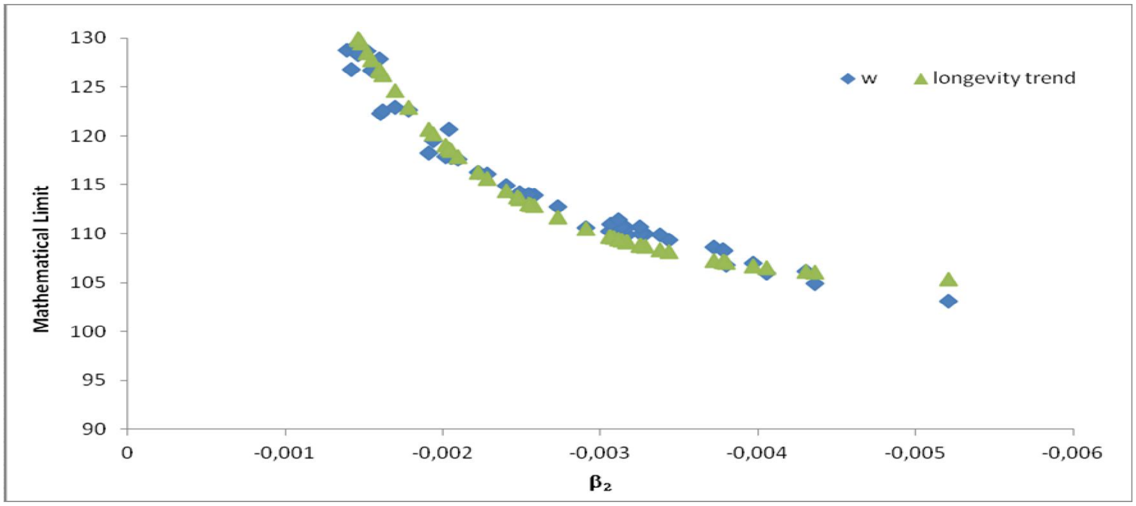

Figure 8: Longevity trend

In Figure 8 you can see that the quadratic relation between the quadratic coefficient and the w limit of longevity can be well approximated by an exponential-type relation (R^2=98,1%):

$$w \approx 104{,}84 + 113{,}6e^{1031{,}8\beta_2}$$

Represented in the Figure by the green series. Being β_2 negative, the ultimate limit of longevity is obtained for $\beta_2 \to -\infty$ and is set almost equal to 105 years. In fact, if we compare this value with that obtained by Weon [21] and [22] for Italy (116 years for women at 2005, about 125 years for both sexes and for a dataset that goes from 1996 to 2001) we see that the Pasqualitto Model provides a lower estimate of this limit and, in our opinion, more in line with those that are realities objectively verifiable on real data. You must moreover consider that the demographic tables used in our analysis refer to the total population, both males and females, and there is no doubt that the higher male mortality and the wider time horizon under consideration necessarily affect the result with downward values.

In support of this analysis, in Appendix 1 we reported a list of ultra centenarians in Italy, namely a list of the number of people over 110 years old (Source: Wikipedia updated 14.02.2012). Indeed it follows that to date only 106 people are actually over 110 years old (of which only 6 are still alive: 5 women and 1 man) and this concerns individuals born between 1860 and 1901, of which 98 women and only 8 men.

We also point out how the extreme value of longevity however roams around an average value just over 110 years, confirming that, in any case, are not recognizable extreme values of longevity however exceeding that amount. Having moreover taken cognizance in our analysis that the limit of longevity has gradually decreased over time, we must point out that demographic changes are however rather slow by nature, such as the recognition of possible trends requires long periods rather than short ones. For all these considerations it therefore seems reasonable the extreme value of about 105 years achieved in our work as ultimate limit of longevity, since we referred to time series for the Italian population starting from 1950, that is, about a century after the first ultra centenarian reported in the appendix, who in fact was born in 1860.

Results

The evolution of longevity is behind a number of actuarial calculations and cannot be underestimated for the economic consequences that may arise within the several insurance companies as a result of an improper and not adequate estimate of the phenomenon itself. Referring to the methodology used by Weon for the search of the limit of longevity, our goal was to identify a new function that would approximate that of survival at older ages, ages that have always been demographically critical to be represented and synthesized by an appropriate model. Starting from the analysis and conclusions reached by Weon in his works, we wanted to carry out a more detailed analysis of the Italian situation.

In particular, we referred to data provided by the Human Mortality Database website (www.mortality.org) for the Italian population (both sexes) from 1950 to 2006. The length of the time series under consideration is the result of the decision to avoid that our analysis could be affected by anomalous values such as those produced by the world wars on mortality.

Considering this time series we have noted, through numerical analysis and simulation exercises that the function S (x) produced by Weon, while adapting well to the characteristics of our population, could still be susceptible to further improvements especially with reference to older ages. Our analysis led us to the function (approximating that of survival) given by the following relation:

$$S(x) = \frac{ke^{\left[-\left(\frac{x}{\alpha}\right)\right]^{\beta(x)}}}{1 + ce^{\left[-\left(\frac{x}{\alpha}\right)\right]^{\beta(x)}}}$$

In which it is clear that the same is completely defined by the value of only two parameters: α called characteristic life and $\beta(x)$, called shape parameter which, unlike traditional models, is no longer constant but age-dependent and provides a measure of the rectangularization that the survival function has had over time. Through appropriate passages we have seen that for ages x> α the shape parameter can be approximated by a quadratic relation and was just the knowledge of the quadratic coefficient of this relation, β_2, to provide us with a useful indication about the existence or not of an ultimate limit of longevity. In particular, the performed analysis has shown how the ultimate w limit of longevity can be well approximated by an exponential-type relationship defined as follows (R^2=98, 1%):

$$w \approx 104{,}84 + 113{,}6e^{1031{,}8\beta_2}$$

Finally, being β_2 negative, the ultimate value of the mathematical limit of longevity is obtained for $\beta_2 \to -\infty$ and is set approximately equal to 105 years. In fact, if we compare this value with that obtained by Weon [21] and [22] for Italy (116 years for women at 2005, about 125 years for both sexes and for a dataset that goes from 1996 to 2001) we see that the Pasqualitto Model provides a lower estimate of this limit and, in our opinion, more in line with those that are realities objectively verifiable on real data.

You must moreover consider that the demographic tables used in our analysis refer to the total population, both males and females, and there is no doubt that the higher male mortality and the wider time horizon under consideration necessarily affect the result with downward values. In support of this analysis, in Appendix 1 we reported a list of ultra centenarians in Italy, namely a list of the number of people over 110 years old (Source: Wikipedia updated 14.02.2012).

Indeed it follows that to date only 106 people are actually over 110 years old (of which only 6 are still alive: 5 women and 1 man) and this concerns individuals born between 1860 and 1901, of which 98 women and only 8 men.

We also point out how the extreme value of longevity however roams around an average value just over 110 years, confirming that, in any case, are not recognizable extreme values of longevity however exceeding that amount. Having moreover taken cognizance in our analysis that the limit of longevity has gradually decreased over time, we must point out that demographic changes are however rather slow by nature, such as the recognition of possible trends requires long periods rather than short ones. For all these considerations it therefore seems reasonable the extreme value of about 105 years achieved in our work as ultimate limit of longevity.

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APPENDIX 1

N.	Sex	Date of birth (dd/mm/yyyy)	Date of death(dd/mm/yy)	Years e Days
1	F	16/09/1860	23/06/71	110 Years, 280 Days
2	F	19/03/1863	05/04/73	110 Years, 17 Days
3	F	30/04/1867	17/04/78	110 Years, 352 Days
4	F	14/05/1875	03/05/86	110 Years, 354 Days
5	M	10/05/1880	22/05/91	111 Years, 12 Days
6	F	7/10/1880	20/02/91	110 Years, 136 Days
7	F	17 /10/1880	10/01/91	110 Years, 85 Days
8	F	1 /03/ 1881	06/03/92	111 Years, 5 Days
9	F	14 /06/ 1881	12/11/91	110 Years, 151 Days
10	F	1º /08/ 1882	07/01/93	110 Years, 159 Days
11	F	12 /09/ 1883	16/10/93	110 Years, 34 Days
12	F	11 /10/ 1883	10/12/95	112 Years, 60 Days
13	F	8 /08/ 1885	04/10/95	110 Years, 57 Days
14	M	2 /01/1886	01/01/97	110 Years, 365 Days
15	F	9 /07/ 1886	06/06/97	110 Years, 332 Days
16	F	1 /04/ 1887	06/01/98	110 Years, 280 Days
17	M	6 /07/ 1887	28/08/98	111 Years, 53 Days
18	F	10 /08/ 1887	01/12/97	110 Years, 113 Days
19	F	11 /10/ 1887	16/06/98	110 Years, 248 Days
20	F	28 /05/ 1888	16/07/98	110 Years, 49 Days
21	F	19 /11/ 1888	04/02/99	110 Years, 77 Days
22	M	22 /01/1889	03/01/02	112 Years, 346 Days
23	F	20 /02/ 1889	21/05/99	110 Years, 90 Days
24	F	2 /12/ 1889	14/05/03	113 Years, 163 Days
25	F	8 /12/ 1889	31/01/00	110 Years, 54 Days
26	F	25 /12/ 1889	07/01/00	110 Years, 13 Days
27	F	13 /02/ 1890	25/05/01	111 Years, 101 Days
28	F	30 /04/ 1890	05/02/01	110 Years, 281 Days
29	M	29 /12/ 1890	19/06/03	112 Years, 172 Days
30	F	6 /07/ 1891	04/01/02	110 Years, 182 Days
31	F	19 /11/ 1891	09/01/02	110 Years, 51 Days
32	F	28 /11/ 1891	06/01/02	110 Years, 39 Days
33	M	8 /12/ 1891	01/04/02	110 Years, 114 Days
34	F	24 /12/ 1891	28/12/05	114 Years, 4 Days
35	F	10 /01/1892	23/02/03	111 Years, 44 Days
36	F	26 /02/ 1892	20/05/03	111 Years, 83 Days
37	F	29 /03/ 1892	02/01/05	112 Years, 279 Days
38	F	2 /08/ 1892	18/09/03	111 Years, 47 Days
39	F	15 /09/ 1892	25/12/02	110 Years, 101 Days
40	F	2 /12/ 1892	14/08/04	111 Years, 256 Days
41	F	12 /02/ 1893	06/02/04	110 Years, 359 Days
42	F	9 /06/ 1893	01/01/05	111 Years, 206 Days
43	F	24 /07/ 1893	13/10/03	110 Years, 81 Days
44	F	15 /09/ 1893	04/09/06	112 Years, 354 Days
45	F	10 /10/ 1893	29/03/06	112 Years, 170 Days
46	F	13 /10/ 1893	29/11/04	111 Years, 47 Days
47	M	12 /11/ 1893	16/03/04	110 Years, 125 Days
48	F	12 /05/ 1894	04/11/05	111 Years, 176 Days
49	F	17 /05/ 1894	29/03/05	110 Years, 316 Days
50	F	19 /06/ 1894	16/01/05	110 Years, 211 Days
51	F	7 /08/ 1894	11/01/07	112 Years, 157 Days
52	F	1 /10/ 1894	30/04/05	110 Years, 211 Days
53	M	23 /12/ 1894	22/01/05	110 Years, 30 Days
54	F	23 /02/ 1895	03/04/05	110 Years, 39 Days
55	F	12 /03/ 1895	13/01/07	111 Years, 307 Days
56	F	12 /06/ 1895	16/09/05	110 Years, 96 Days
57	F	19 /06/ 1895	11/05/06	110 Years, 326 Days
58	F	10 /07/ 1895	15/01/06	110 Years, 189 Days
59	F	20 /01/1896	30/01/06	110 Years, 10 Days
60	F	4 /03/ 1896	28/06/09	113 Years, 116 Days
61	F	1 /07/ 1896	15/03/07	110 Years, 257 Days
62	F	16 /08/ 1896	12/03/07	110 Years, 210 Days
63	F	4 /10/ 1896	02/11/09	113 Years, 29 Days
64	F	3 /11/ 1896	26/12/07	111 Years, 53 Days
65	F	22 /11/ 1896	27/05/09	112 Years, 186 Days
66	F	23 /11/ 1896	02/08/11	114 Years, 252 Days

67	F	6 /12/ 1896	12/01/07	110 Years, 37 Days
68	F	11 /02/ 1897	02/11/08	111 Years, 265 Days
69	F	10 /06/ 1897	02/09/07	110 Years, 84 Days
70	F	1 /08/ 1897	02/05/08	110 Years, 275 Days
71	F	27 /09/ 1897	15/11/07	110 Years, 49 Days
72	F	19 /10/ 1897	06/04/08	110 Years, 170 Days
73	F	21 /10/ 1897	15/06/09	111 Years, 237 Days
74	F	15 /01/1898	05/04/08	110 Years, 81 Days
75	F	12 /02/ 1898	04/07/08	110 Years, 140 Days
76	F	21 /02/ 1898	11/04/08	110 Years, 50 Days
77	F	3 /04/ 1898	03/05/08	110 Years, 30 Days
78	F	1 /06/ 1898	24/04/09	110 Years, 327 Days
79	F	12 /08/ 1898	16/03/09	110 Years, 216 Days
80	M	23 /08/ 1898	26/10/08	110 years, 64 days
81	F	15 /11/ 1898	11/01/10	111 years, 57 days
82	F	23 /12/ 1898	alive	113 years e 53 days
83	F	23 /02/ 1899	18/06/11	112 years, 115 days
84	F	3 /04/ 1899	alive	112 e 317 days
85	F	5 /04/ 1899	20/09/09	110 years, 168 days
86	F	21 /06/ 1899	28/02/11	111 years, 252 days
87	F	21 /07/ 1899	17/08/11	112 years, 27 days
88	F	11 /08/ 1899	24/06/10	110 years, 317 days
89	F	7 /10/ 1899	29/03/10	110 years, 173 days
90	F	27 /11/ 1899	18/02/11	111 years, 83 days
91	F	29 /11/ 1899	alive	112 years e 77 days
92	F	25/01/1900	02/01/11	110 years, 342 days
93	F	11/02/1900	07/04/11	111 years, 55 days
94	F	03/03/1900	alive	111 years e 348 days
95	F	03/05/1900	27/12/11	111 years, 238 days
96	F	28/06/1900	12/01/11	110 years, 168 days
97	F	16/08/1900	03/06/11	110 years, 260 days
98	F	02/10/1900	09/10/10	110 years, 7 days
99	F	21/10/1900	03/07/11	110 years, 255 days
100	F	19/11/1900	23/01/12	111 years, 65 days
101	F	08/12/1900	18/06/11	110 years, 192 days
102	F	14/02/1901	22/08/11	110 years, 189 days
103	M	18/02/1901	alive	110 years e 361 days
104	F	04/06/1901	alive	110 years e 255 days
105	F	24/06/1901	30/08/11	110 years, 67 days
106	F	20/07/1901	09/08/11	110 years, 20 days

Chart1: People over 110 years in Italy, Source Wikipedia updated to 14/12/2012