Modeling Birdmen: An Example of Mathematical Modeling with a Linear Differential Equation

Dr. Jane Whitmire
Central Washington University
United States of America

When a person jumps off a cliff, they will certainly feel the effects of gravity and so will be accelerated downwards. However, they will also feel wind resistance. This paper mathematically models the flight of the Birdmen as presented in a CBS video on October 11, 2009. The reader can watch the 60 Minutes video presented by Steve Kroft on the website: <http://www.cbsnews.com> using key word: Birdmen.

Birdmen like to jump off extremely steep cliffs. Air inflates the wings of their nylon suit and propels the Birdman forward. Eventually, a Birdman will reach some constant velocity. This is the moment that one might imagine the feeling of control over flight begins. If planning to jump, we might take some comfort in knowing how long it takes to gain this sense of well-being.

To determine how many seconds until a constant velocity, we begin with a differential equation that models a free-falling object. If $v(t)$ represents the vertical velocity, a low resolution model for this situation is given by;

$$\frac{dv}{dt} = g - kv$$

Here, g is constant describing acceleration due to gravity and k is a positive constant representing the force of friction. Notice that when the velocity v is zero, the acceleration simplifies to gravitational force:

$$\frac{dv}{dt} = g$$

Thus, as the velocity increases, the acceleration will decrease. We can solve this differential equation using an integrating factor $e^{\int kdt}$ to get the velocity per time equation.

$$\frac{dv}{dt} = g - kv$$

$$\frac{dv}{dt} + kv = g$$

$$\mu = e^{\int kdt} = e^{kt}$$

To find the solution to this differential equation, we simply multiply the initial equation by the integrating factor, recognize the implicit form of the derivative on the left hand side, and integrate.

$$\begin{aligned} e^{kt}dv + kve^{kt}dt &= ge^{kt}dt \\ d[ve^{kt}] &= ge^{kt}dt \\ ve^{kt} &= \int ge^{kt}dt + c_1 \end{aligned}$$

$$
\begin{aligned}
ve^{kt} &= \frac{g}{k}e^{kt} + c_2 \\
v(t) &= \frac{g}{k} + c_2 e^{-kt}
\end{aligned}
$$

Acceleration is the derivative of the velocity function. The acceleration will be zero when the velocity is equal to g/k.

$$
\begin{aligned}
a(t) &= \frac{dv}{dt} \\
&= g - kv
\end{aligned}
$$

$$
\begin{aligned}
g - kv &= 0 \\
-kv &= -g \\
v &= \frac{g}{k}
\end{aligned}
$$

Thus, we have shown that the acceleration is given by $a(t) = g - kv$ and the velocity is given by $v(t) = \frac{g}{k} + c_2 e^{-kt}$.

So what happens to the velocity as time goes by? Describing what happens when t is very large is equivalent to interpreting the limit of $v(t)$ as $t \to \infty$.

$$
\lim_{t \to \infty}\left(\frac{g}{k} + c_2 e^{-kt}\right)
$$

Consider what happens to the term, $c_2 e^{-kt}$ as t gets very large and c_2 is held constant. We see that the second term, $c_2 e^{-kt}$ goes to zero and the velocity, *v(t),* becomes a constant: *g/k*. Thus, when a person free-falls, the downward velocity will eventually level off to a constant. How quickly this happens is dependent on the values of c_2 and *k*. The following graph depicts the scenario for $c_2 = 1$ and *k=1,3,5.*

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In[11]:= Plot[{y = Exp[-k], y[k] = Exp[-3 k], y[k] = Exp[-5 k]}, {k, 0, 10}]
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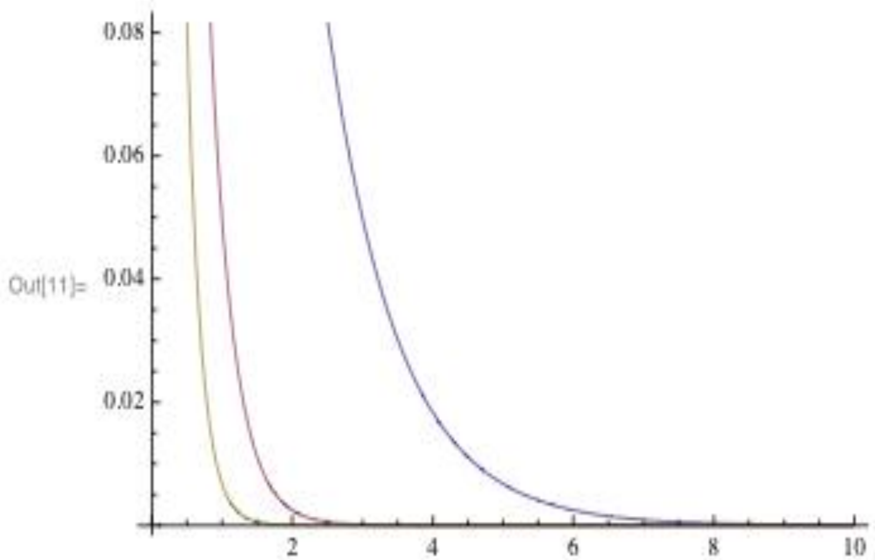


In the video, the Birdman said that he was propelled forward 2 feet for each foot that he fell. He estimated his horizontal speed to be between 140 and 150 miles per hour. Thus, his vertical velocity leveled off to about 75 miles per hour.

If we adopt English units for gravity, 32 feet per second squared, there is enough information to calculate the value of *k*.

$$v = \frac{g}{k} = \frac{75 miles}{hour} = \frac{110 feet}{\sec ond}$$

$$\frac{g}{k} = \frac{\left(32\, feet \big/ \sec^2\right)}{k} = \frac{110\ feet}{\sec}$$

This leads to approximately, *k=0.2909/sec* and our model becomes;

$$v(t) = 110\frac{feet}{\sec} + c_2 e^{-\frac{0.2909}{\sec}t}$$

To solve for c_2 observe that at time *t=0* the initial velocity is zero. Therefore,

$$c_2 = -110\frac{feet}{\sec}$$

We can use the values of *k* and c_2 to update our velocity equation. This equation models the velocity of the flight of the Birdman using the details he provided in the CBS interview.

$$v(t) = \left(110\frac{feet}{\sec}\right)\left(1 - e^{-\frac{0.2909}{\sec}t}\right)$$

Let's go back to the assumption that we had the nerve to try this out. ``How many seconds does one fall, with non-zero acceleration, in flight, before velocity becomes constant?" The surface area of the wings will create air friction. So a constant downward velocity must happen reasonably soon for a Birdman?

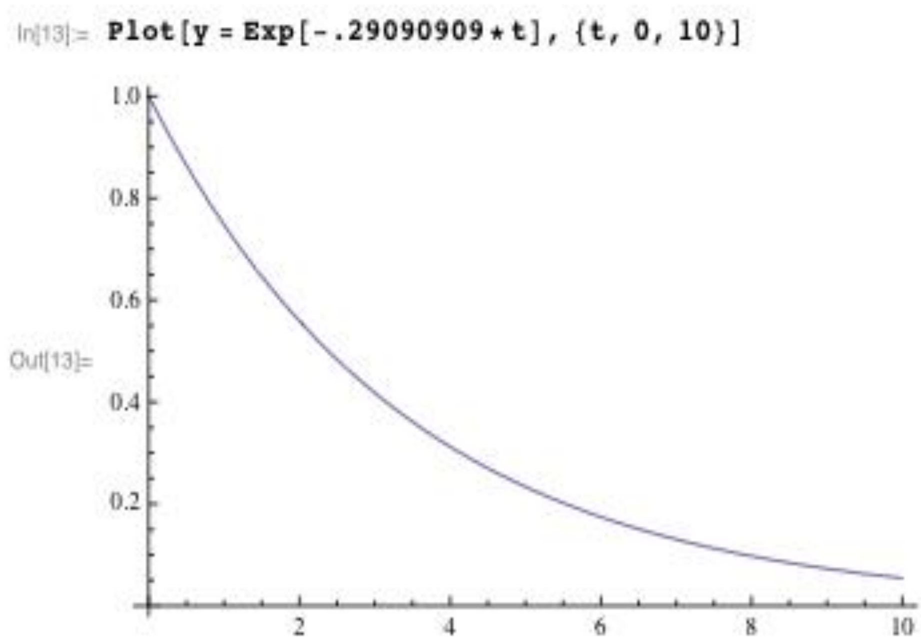
We seek the time, t, for which the downward velocity, $v(t)$ is constant at about 75 miles per hour or equivalently 110 feet per second. Find this by setting the velocity function equal to 110 feet per second.

$$\left(110\frac{feet}{\sec}\right)\left(1-e^{-\frac{0.2909}{\sec}t}\right) = 110\frac{feet}{\sec}$$

$$\left(1-e^{-\frac{0.2909}{\sec}t}\right) = 1$$

$$e^{-\frac{0.2909}{\sec}t} = 0$$

The final equation is not true for any set value of t, but can be approximated by looking at values of the t where the graph of $y = e^{-\frac{0.2909}{\sec}t}$ approaches zero.



We see that it takes about 10 seconds before the Birdman is flying at a relatively constant velocity. The sensation of gaining control happens gradually during the first few seconds of flight. This paper, however, justifies the decision not to jump!

References

60 Minutes, {\it Birdmen} hosted by Steve Kroft, www.cbsnews.com

Images provided with the use of Wolfram Alpha, www.wolframalpha.com